Relational Propositional Logic.

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Contents

Relational propositional logic differs from classic by 3 features.

First, relational logic is not the calculus of valid formulas. It is the science of construction of the zero order theories, i.e. theories which objects are propositions. So the axiom system is changed.

Second, propositional logic belongs to the first order theory. So formulas of the zero order theories are presented by means of proposional logic, but formulas of propositional logic are presented by means of the first order logic. Therefore we use different means to state objects of propositional logic and state propositional logic itself.

Third, all zero order theories are presented in a computer-oriented form.

Computer-oriented form differs from classic by absence of brackets and connectives. For this we use conjunctive normal form (CNF) to present any formulas as a set of clauses and any clause as a set of literals. We suppose that members of the clause set are connected by conjunctions and members of the literal set are connected by disjunctions. Any *literal* is an *atom* (propositional symbol) without or with negation. In the first case the literal is positive, in the second case the literal is negative.

In the computer-oriented form valid formulas are absent. But other formulas have one-one mapping of classic and computer-oriented forms.

So the classic form of formula presentations is main to state base of propositional logic because this form is more usual and clear.

Below we use two ways to construct propositional logic.

In section 1 we use the syntactic way. We introduce a mathematical logic language for the syntactic definition of logical formulas. This definition allows to extract a formula subset from a word set for a given alphabet. Then we construct a set of all theories using these formulas.

This way is just theoretical. It can not be executed because of the problem of the large numbers. In particular we must construct $3 \cdot 10^{102}$ consistent theories from 10 atoms. But we constructed and investigated all theories from 5 or less atoms.

In further sections we use the computer-oriented way to construct axiom calculus for a theory.

In these sections we give rules to verify consistency or inconsistency of theories. These rules have high effectiveness and their time dependence is quadratic as limit but quasi-linear for most problems.

The rule of theory inconsistency are basic in theorem proving. Every consistent theory becomes inconsistent if we add negation of any theorem to the theory. So we must add negation of a proved theorem to axioms of the theory and then prove inconsistency of a new axiom system.

We give also the rule for the logical closure of theories. i.e. for construction of all the theorems of the theory. Besides, we give the rule to construct all independent axiom systems in a theory. So we can find the system with a minimal number of axioms or with minimal sum of axiom lengths. Both rules are highly effective.

1. Mathematical Logic Language. Incompleteness of Logic.

The propositional logic signature $< S, C, F, R >$ consists of sort set S with one member, empty sets of constants C and function symbols F and a countable set of 0-place relational symbols R.

Arbitrary members of R we shall denote as R_0, R_1, R_2, ...

The arbitrary members are meta-variables. These meta-variables can be replaced with concrete members by substitutions. The concrete members we shall denote A, B, C, ...

The alphabet of propositional logic includes all members of R, parenthesis and logical symbols of implication "$\rightarrow$" and negation "$\neg$". All words must be finite.

We denote W the set of all words in the alphabet and W_0, W_1, W_2, ... arbitrary members of this set. The arbitrary member is meta-variable and belongs to meta-logic.

Meta-logic is in fact the first order logic. It is natural because propositional logic belongs to the set of the first order theories.

We must define the recursive relation $\mathcal{F}$ "to be a formula".

Definition 1.1. The relation $\mathcal{F}(W_0)$ means "the word W_0 is a *formula* of propositional logic", if

$$\mathcal{F}(W_0)\dot{\rightleftarrows}(\exists R_0\ W_0 = R_0)\dot{\vee}(\exists W_1 \exists W_2\ W_0 = (\mathcal{F}(W_1) \rightarrow \mathcal{F}(W_2)))\dot{\vee}\exists W_1\ W_0 = (\neg\mathcal{F}(W_1)).$$

This definition uses meta-logic. Logical connectives of meta-logic (except quantifiers) are marked by the dot above.

The given definition is divided into two ones. Replacing equivalence by back implication we have the positive definition. After reducing to CNF and using equality properties we get:

$\mathcal{F}(R_0)$.

$\mathcal{F}(\quad (\mathcal{F}(W_1) \rightarrow \mathcal{F}(W_2)) \quad)$.

$\mathcal{F}(\ (\neg\mathcal{F}(W_1))\ )$.

So we have the definition of the formula by induction. The formula is:

R_0 (an atomic formula, i.e. an atom);

$(W_1 \rightarrow W_2)$, if W_1 and W_2 are formulas;

$(\neg W_1)$, if W_1 is a formula.

Therefore the symbol R_0 is simultaneously an atom.

The negative definition is got if equivalence is replaced by usual implication:

$$\mathcal{F}(W_0)\dot{\rightarrow}\exists R_0\ W_0 = R_0\dot{\vee}\exists W_1\exists W_2\ W_0 = (\mathcal{F}(W_1) \rightarrow \mathcal{F}(W_2))\dot{\vee}\exists W_1\ W_0 = (\neg\mathcal{F}(W_1)).$$

We get the final part of the inductive definition:

The other formulas do not exist.

We shall denote $\mathcal{F}$ the set of all formulas and $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$, ... arbitrary members of this set.

In reality we define the sort $\mathcal{F}$ interpreted as a formula set. The variables of this sort $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$,... are interpreted as arbitrary members of $\mathcal{F}$.

Definition 1.2. The formulas F_1 and F_2 are called *congruous* if there exists a substitution to transform one formula in other.

We means the substitution one atom by other because atoms are pseudo-variables.

The special formulas are valid and inconsistent.

Definition 1.3. The formula F_0 is *valid* if it satisfies the relation $\mathcal{V}(\mathcal{F}_0)$. Below we give only the positive definition of this relation:

$$\mathcal{V}(\quad (\mathcal{F}_1 \rightarrow (\mathcal{F}_2 \rightarrow \mathcal{F}_1)) \quad).$$
$$\mathcal{V}(\quad ((\mathcal{F}_1 \rightarrow (\mathcal{F}_2 \rightarrow \mathcal{F}_3)) \rightarrow ((\mathcal{F}_1 \rightarrow \mathcal{F}_2) \rightarrow (\mathcal{F}_1 \rightarrow \mathcal{F}_3))) \quad).$$
$$\mathcal{V}(\quad (((\neg\mathcal{F}_1) \rightarrow (\neg\mathcal{F}_2)) \rightarrow (((\neg\mathcal{F}_1) \rightarrow \mathcal{F}_2) \rightarrow \mathcal{F}_1)) \quad).$$
$$\mathcal{V}(\mathcal{F}_1)\dot{\wedge}\mathcal{V}(\quad (\mathcal{F}_1 \rightarrow \mathcal{F}_0) \quad)\dot{\rightarrow}\mathcal{V}(\mathcal{F}_0).$$

The first three meta-formulas are called axioms and the last meta-formula is called the inference rule of propositional logic. The negative definition is ignored as a rule.

All these are not true. Logic is not the calculus of valid formulas but is a science of theory construction with its own axioms and inference rule. Below we shall show this science consists of theory and axiom calculi.

Definition 1.4. Formula $\mathcal{F}_0$ is *inconsistent* $\mathcal{C}(\mathcal{F}_0)$ if it is negation of a valid formula:

$$\mathcal{C}(F_0)\dot{\rightleftarrows}\mathcal{V}(\neg F_0).$$

We must delete equivalent and congruous formulas from $\mathcal{F}$. This deletion will be realized below.

Definition 1.5. The *logic equivalent formulas* $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy the relation $\mathcal{E}(\mathcal{F}_1, \mathcal{F}_2)$:

$$\mathcal{E}(\mathcal{F}_1, \mathcal{F}_2) \rightleftarrows \mathcal{V}((\mathcal{F}_1 \rightleftarrows \mathcal{F}_2))$$

where the second symbol of equivalence belongs to proposional logic.

So two formulas are logical equivalent if their equivalence is a valid formula.

Now we introduce denotations for negation, conjunction, disjunction and equivalence:

$(\overline{\mathcal{F}}_1) \rightleftharpoons (\neg\mathcal{F}_1)$

$(\mathcal{F}_1, \mathcal{F}_2) \rightleftharpoons (\neg(\mathcal{F}_1 \to (\neg\mathcal{F}_2)))$

$(\mathcal{F}_1\mathcal{F}_2) \rightleftharpoons ((\neg\mathcal{F}_1) \to \mathcal{F}_2)$

$(\mathcal{F}_1 \rightleftarrows \mathcal{F}_2) \rightleftharpoons ((\mathcal{F}_1 \to \mathcal{F}_2), (\mathcal{F}_2 \to \mathcal{F}_1))$

Example 1.7. We demonstrate the use of connectives:

$$A\ B\ C,\ \overline{A}\,\overline{B}\,\overline{C}$$

In a more close record we have

$$ABC, \overline{ABC}$$

This example is more extended, if we use traditional notions:

$$(A \lor B \lor C) \land (\neg A \lor \neg B \lor \neg C) \quad \square$$

Now we can distinguish the symbols of logic and meta-logic. The exceptions are the symbols of comma, implication, and equivalence. Below the symbol of equivalence is used in metalogic, if another is not mentioned. The symbol of implication is not used in logic as a rule. The symbol of comma does not lead to misunderstanding because an operation with comma is taken in additional parenthesis in mixed formulas.

We must define the conjunctive normal form CNF.

Definition 1.8. The *clause* $C(W_1)$ is a word W_1 consisting of literals (joined by disjunctions). The clause has no coinciding atoms.

Definition 1.9. The *conjunctive normal form* CNF(W_1) is a word W_1 being the sequence of clauses separated by conjunctions. The CNF has no coinciding clauses (coinciding clauses consist of the same literals).

The particular case of CNF is the perfect conjunctive normal form PCNF. The form is necessary for theory calculus and we must define this form.

Definition 1.10. The *perfect conjunctive normal form* PCNF(W_1) is CNF which all clauses consist of the same atoms.

All clauses of PCNF consist of the same atoms, but not the same literals. A number of literals in every clause is equal.

Definition 1.11. The theory is *perfect PT* if all axioms of this theory are clauses and consist of the same atoms.

A number of literals in each axiom equals a number of atoms in perfect theory. Joining axioms of a perfect theory by conjunctions we get PCNF.

Two perfect theories are equal if they consist of the same clauses.

Two perfect theories are *congruous* if there exist atom substitutions to transform one theory to another.

We shall investigate a perfect theory constructed with the same number of atoms. Because of congruence all these theories can be constructed from the first n members of set R, where n equals a number of atoms in the theory.

Below we shall see an arbitrary theory with n atoms to be equivalent to some perfect theory with the same number of atoms.

So, we can limit ourselves by the perfect theories. Theory calculus lets us construct all the theories with n atoms. Increasing n we can construct all the theories with a different number of atoms.

Therefore the theory calculus can be limited by the construction of all perfect theories with n atoms.

Definition 1.12. The *theory calculus* Th^n with n atoms is set PT^n of all perfect theories containing the first n members from the set R:

$$Th^n = PT^n$$

The set PT^n is constructed by using the definition of PT.

Constructing Th^n we create all clauses containing the first n members of R. A number of these clauses m equals 2^n. Then we take all combination of m clauses i at a time $(0 < i < m)$. If $i = 0$ then we have the empty theory (note reducing a valid formula to PCNF this formula becomes empty). If $i = m$ then PCNF becomes inconsistent. The other PCNFs are neither valid nor inconsistent formulas. So the total number of these PCNFs are $2^{2^n} - 2$. Below we use only that PCNF.

One more calculus is the axiom one. This calculus will be stated in details in the next section. Here we shall use axiom construction to compare theories.

We can compare theories if we construct the logical closed theory. Below we call these theories just closed.

Definition 1.13. The *theory closure* is addition all the theorems in this theory, besides subsumed theorem, to the axiom set. The resulting theory is called *closed*. The theorems are clauses and so they are called axioms too.

Definition 1.14. A clause is *subsumed* if it contains all the literals of another clause.

Subsumed clause have, as a rule, superfluous literals. So they are unnecessary.

Definition 1.15. The *reduced* CNF is joint of closed theory clauses by conjunctions.

Since every theory becomes CNF if all clauses of this theory are joined by conjunction. Otherwise every CNF becomes theory after removing all conjunctions.

Theorem 1.16. *Two theories are logical equivalent if their closures are equal.*

Proof: Theorems are deduced by the inference rule. But this rule is a valid formula. It means theories with and without theorems are logical equivalent (LE). So a theory and its closure are LE. If two theories have the same closure, they are LE because logical equivalence is transitive. □

Note the closure of a inconsistent theory is empty.

Now we return again to the zero order theory calculus. We shall not use meta-logic notation for connectives till the end of this section. So we can use propositional logic notation introduced above and usual notation too.

Let n be a number of atoms for theory construction.

If $n = 1$ then there exist two theories:

$$\overline{A} \quad A.$$

Standard interpretations of these theories are one-place connectives "negation" and "double negation" ("empty") with truth tables:

$\overline{A}$

$\neg$	
t	f
f	t

A

$\not\neg$	
t	t
f	f

Both theories are automorphic over a literal set $\{A, \overline{A}\}$: we get the first theory from the second to replace A by $\overline{A}$ and on the contrary we get the second theory from the first to replace $\overline{A}$ by A. So we can use only one of these theories, for example, the first:

$$\overline{A}. \qquad (T1)$$

Definition 1.17. Two theories are *automorphic* over a literal set, if there exist substitutions of literals and after using substitutions to the first theory we get the second one. After using back substitution to the second theory we get the first one.

Below we use the term "automorphism" instead of the term "automorphism over a literal set" for simplification.

If $n = 2$ then there are 14 perfect theories and correspondingly 14 closed theories:

$A.\ \ \overline{A}.\ \ B.\ \ \overline{B}.\ \ A, B.\ \ A, \overline{B}.\ \ \overline{A}, B.\ \ \overline{A}, \overline{B}.\ \ AB.\ \ A\overline{B}.\ \ \overline{A}B.\ \ \overline{AB}.\ \ AB, \overline{AB}.\ \ A\overline{B}, \overline{A}B.$

These 14 theories have 14 truth tables:

A

	t	f
t	t	t
f	f	f

$\overline{A}$

	t	f
t	f	f
f	t	t

B

	t	f
t	t	f
f	t	f

$\overline{B}$

	t	f
t	t	f
f	t	f

A, B

$\wedge$	t	f
t	t	f
f	f	f

$A, \overline{B}$

$\not\rightarrow$	t	f
t	f	t
f	f	f

$\overline{A}, B$

$\not\leftarrow$	t	f
t	f	f
f	t	f

$\overline{A},\overline{B}$

$\not\vee$	t	f
t	f	f
f	f	t

AB

$\vee$	t	f
t	t	t
f	t	f

$A\overline{B}$

$\leftarrow$	t	f
t	t	t
f	f	t

$\overline{A}B$

$\rightarrow$	t	f
t	t	f
f	t	t

$\overline{AB}$

$\not\wedge$	t	f
t	f	t
f	t	t

$AB,\overline{AB}$

$\not\rightleftarrows$	t	f
t	f	t
f	t	f

$A\overline{B},\overline{A}B$

$\rightleftarrows$	t	f
t	t	f
f	f	t

Removing automorphic theories except one their representative we have 4 theories:

$$\overline{A}.\quad A,B.\quad AB.\quad A\overline{B},\overline{A}B.$$

These theories correspond to connectives:

$$\neg\quad \wedge\quad \vee\quad \rightleftarrows$$

Definition 1.18. The theory over a set of n atoms is *degenerate* if this theory has only part of this atom set.

Degenerate theories are superfluous. After removing them we have 3 theories. Below we shall see theories with logical equivalence to be superfluous too. So we have only two theories:

$$A,B.\quad AB. \tag{$T2$}$$

Therefore we have only 3 connectives, one from ($T1$) and two from ($T2$).

Thus automorphism (and equivalence) reduces a number of connectives to 3.

It is known that function superposition reduces a number of connectives to one, but all investigators prefer to use the first case.

If $n = 3$ a number of theories becomes 254. Removing automorphic we have 20 theories, 4 of them are degenerate. Among others we have 8 closed theories with logical equivalence:

$$A(B \rightleftarrows \overline{C}).\quad A(B \rightleftarrows C),\overline{A}BC.\quad A \rightleftarrows \overline{BC}.\quad A(B \rightleftarrows \overline{C}),\overline{A}(B \rightleftarrows C).$$
$$A \rightleftarrows \overline{BC},BC.\quad A \rightleftarrows B,AC,BC.\quad A,B \rightleftarrows \overline{C}.\quad A \rightleftarrows \overline{B},A \rightleftarrows \overline{C},B \rightleftarrows C.$$

We shall show these theory are superfluous.

It is known all axioms are divided into independent, dependent (theorems) and definitions. The last have logical equivalence between a defined object and a defining formula.

Recursive definitions are absent in the zero order theories. Hence all definitions are used to simplify complex formulas.

Therefore we can remove any definition to replace a defined object by a defining formula. We can remove definitions too when a defined object is absent in remaining axioms. This definition is just superfluous.

After we remove axioms with equivalencies the theory coincides with one of other theories to within automorphism.

Example 1.19. The theory $A(B \rightleftarrows C),\overline{A}BC$ becomes $\overline{A}BC$ after the definition $A(B \rightleftarrows C)$ is removed. This theory coincides with the theory ABC within automorphism. The theory ABC is presented in ($T3$). □

Thus theories with equivalencies are superfluous.

The other 8 theories are

$$ABC.\ \ ABC,\overline{ABC}.\ \ AB,AC.\ \ AB,\overline{AB}C.\ \ AB,AC,BC.\ \ AB,AC,B\overline{C}.\ \ A,BC.\ \ A,B,C. \tag{$T3$}$$

Definition 1.20. The axiom is *superfluous* if it is not used to deduce theorems.

In particular all axioms are superfluous if their theories have no theorem.

Only one axiom from 8 has no superfluous axiom:

$$AC,B\overline{C} \vdash AB. \tag{$T3'$}$$

Here in the left part (before symbol "⊢") independent axioms are presented, in the right part theorems are presented. In that form all theories without superfluous axioms are presented below. Note this form is a valid formula.

An axiom can not be a valid formula, but a closed theory can be if this theory are present with "⊢". Without this symbol a closed theory is not any formula at all.

If $n = 4$, a number of theories equals 65534. After removing automorphic ones we have 400 theories, 20 of them are degenerate, 266 have equivalencies, 19 have subsumed theorems, and 14 have superfluous axioms.

Subsumed theorems become non-subsumed if we removed subsuming axioms.

These removals are demonstrated in the next table:

	Closed theories	Subsumed theorems
1	$AB, ACD, BC\overline{D}$	$ACD, BC\overline{D} \vdash ABC$
2	$AB, AC, B\overline{C}D$	$AC, B\overline{C}D \vdash ABD$
3	$AB, ACD, BC\overline{D}, \overline{ABC}D$	$ACD, BC\overline{D} \vdash ABC$
4	$AB, AC, AD, BC\overline{D}$	$AD, BC\overline{D} \vdash ABC$
5	$AB, AC, AD, B\overline{CD}$	$AC, B\overline{CD} \vdash AB\overline{D}; AD, B\overline{CD} \vdash AB\overline{C}$
6	$AB, AC, B\overline{C}D, \overline{B}C\overline{D}$	$AB, \overline{B}C\overline{D} \vdash AC\overline{D}; AC, B\overline{C}D \vdash ABD$
7	$AB, AC, \overline{AC}D, B\overline{C}D$	$AC, B\overline{C}D \vdash ABD$
8	$AB, AC, B\overline{C}D, \overline{ABCD}$	$AC, B\overline{C}D \vdash ABD$
9	$AB, ACD, \overline{ABC}, BC\overline{D}$	$ACD, BC\overline{D} \vdash ABC$
10	$AB, AC, AD, BC\overline{D}, \overline{BC}D$	$AB, \overline{BC}D \vdash A\overline{C}D; AC, BC\overline{D} \vdash ABD; AD, BC\overline{D} \vdash ABC$
11	$AB, AD, BC, \overline{A}C\overline{D}$	$AB, \overline{A}C\overline{D} \vdash BC\overline{D}$
12	$AB, AC, B\overline{C}, \overline{AC}D$	$AB, \overline{AC}D \vdash B\overline{C}D$
13	$AB, AC, \overline{AB}D, B\overline{CD}, \overline{B}CD$	$AB, \overline{B}CD \vdash ACD; AC, B\overline{CD} \vdash AB\overline{D}$
14	$AB, AC, AD, BC, \overline{BC}D$	$AB, \overline{BC}D \vdash A\overline{C}D; AC, \overline{BC}D \vdash A\overline{B}D$
15	$AB, AC, AD, B\overline{C}, \overline{B}CD$	$AB, \overline{B}CD \vdash ACD$
16	$AB, AC, AD, B\overline{C}, \overline{B}C\overline{D}$	$AB, \overline{B}C\overline{D} \vdash AC\overline{D}; AD, \overline{B}C\overline{D} \vdash A\overline{B}C$
17	$AB, AC, BD, \overline{AC}D, \overline{B}C\overline{D}$	$AB, \overline{AC}D \vdash B\overline{C}D; AB, \overline{B}C\overline{D} \vdash AC\overline{D}$
18	$AB, AC, B\overline{C}, \overline{AC}D, \overline{B}C\overline{D}$	$AB, \overline{AC}D \vdash B\overline{C}D; AB, \overline{B}C\overline{D} \vdash AC\overline{D}$
19	$AB, AC, B\overline{C}, \overline{AB}D, \overline{AC}D, \overline{B}CD$	$AB, \overline{AC}D \vdash B\overline{C}D; AB, \overline{B}CD \vdash ACD$

As we see, all axioms, creating subsumed theorems, are superfluous, except in the last line.

In the last line we can delete axioms AB and AC or AB and $B\overline{C}$. Then the theory remains closed but without subsumed theorems.

Definition 1.21. The axiom is *passive* if it takes part in deduction but can be removed from the theory and a new theory remains closed.

So an axiom is called passive if after removing this axiom we have again a closed theory. But in this case (line 19) removed axioms are not passive. For example, removing axiom AC we get theory $AB, B\overline{C}, \overline{AB}D, \overline{AC}D, \overline{B}CD$ and this theory is not closed. But removing axioms AB, AC we get theory $B\overline{C}, \overline{AB}D, \overline{AC}D, \overline{B}CD$ and this theory is closed.

We see from the table that every theory with a subsumed theorem has a subtheory without subsumed theorems.

So the theory with subsumed theorems is the wrong extension of another theory. We have analogy: the theory with equivalence is the wrong extension of another theory too.

All the theories with subsumed theorems can be removed because their subtheories coincides with one of remaining theories within automorphism.

After removing theories with subsumed theorems we have 16 theories, but 6 of these theories have passive literals and 6 have passive axioms. The other 4 theories are non-trivial.

Definition 1.22. The literal in an axiom is *passive* if it can be removed from this axiom but other axioms (and theorems) are not changed.

Definition 1.23. The theory is *non-trivial* if it has neither superfluous and passive axioms nor passive literals.

It is interesting to mark that a non-trivial theory has only one independent axiom system.

The other theories have several independent axiom systems. Many of these systems are automorphic. In these theories we removed automorphic systems, except one of their representatives.

Passive literals are in 6 theories:

1. $ABD, AC\overline{D} \vdash ABC$.
2. $ABD, ACDBC\overline{D} \vdash ABC$.
3. $ABC, AC\overline{D}, B\overline{C}D \vdash ABD$.
4. $ABC, A\overline{B}D, \overline{A}B\overline{D} \vdash ACD, BC\overline{D}$.
   $A\overline{B}D, ACD, \overline{A}B\overline{D}, BC\overline{D} \vdash ABC$.
5. $AB, \overline{AB}C, \overline{B}CD \vdash ACD$.
   $AB, ACD, \overline{AB}C \vdash \overline{B}CD$.
6. $AB, \overline{AB}C, \overline{AB}D, \overline{B}CD \vdash ACD$.
   $AB, ACD, \overline{AB}C, \overline{AB}D \vdash \overline{B}CD$.

Example 1.24. Removing the literal C in the axiom $\overline{ABC}$ of theory 6 or the literal D in the axiom $\overline{AB}D$ of the same theory (in last line for both cases) we have again a closed theory. □

Passive axioms are in 6 theories too:

1. $A\overline{B}C, A\overline{BD}, \overline{A}B\overline{C}, BCD \vdash ACD, \overline{A}BD.$
   $A\overline{BD}, ACD, \overline{A}B\overline{C}, BCD \vdash A\overline{B}C, \overline{A}BD.$
2. $AB, \overline{ABD}, \overline{B}CD \vdash ACD, \overline{AB}C.$
   $AB, ACD, \overline{AB}C\overline{ABD} \vdash \overline{B}CD.$
3. $ABC, \overline{ABC}, B\overline{C}D, \overline{B}C\overline{D} \vdash ABD, AC\overline{D}, \overline{ABD}, \overline{AC}D$
   $ABC, \overline{ABD}, \overline{AC}D, B\overline{C}D, \overline{B}C\overline{D} \vdash ABD, AC\overline{D}, \overline{ABC}$ $(T4)$
4. $AB, \overline{AB}C, \overline{AC}D, \overline{B}C\overline{D} \vdash AC\overline{D}, \overline{AB}D, B\overline{C}D$
   $AB, AC\overline{D}, \overline{AB}C, \overline{AC}D \vdash \overline{AB}D, B\overline{C}D, \overline{B}C\overline{D}$
   $AB, AC\overline{D}, \overline{AB}C, \overline{AB}D, B\overline{C}D \vdash \overline{AC}D, \overline{B}C\overline{D}$
5. $AC, BD, \overline{ACD}, \overline{BCD} \vdash A\overline{BD}, \overline{A}B\overline{C}$
   $AC, BD, \overline{A}B\overline{C}, \overline{BCD} \vdash A\overline{BD}, \overline{ACD}$
6. $A\overline{D}, B\overline{C}, CD \vdash AB, AC, BD$

Example 1.25. Theory 3 has 10 independent axiom systems (see example 4.12), but only two of these systems are not automorphic. Every axiom is passive in this theory. Removing the axiom ABC we get the theory with two independent axiom systems:

$$ABD, AC\overline{D}, \overline{ABC}, B\overline{C}D, \overline{B}C\overline{D} \vdash \overline{ABD}, \overline{AC}D.$$
$$ABD, AC\overline{D}, \overline{ABD}, \overline{AC}D, B\overline{C}D, \overline{B}C\overline{D} \vdash \overline{ABC}.$$

In this theory axioms ABD,$AC\overline{D}$,$\overline{ABC}$ are passive.

Removing the axiom ABD we get the theory with one independent axiom system:

$$AC\overline{D}, \overline{ABC}, B\overline{C}D, \overline{ABD} \vdash \overline{AC}D, \overline{B}C\overline{D}.$$

In the new theory we have two passive axioms $AC\overline{D}$ and $B\overline{C}D$. After removing one of them we have the other axiom passive. But removing both axioms we come back to the theory with two independent axiom systems:

$$\overline{ABC}, \overline{AC}D, \overline{B}C\overline{D} \vdash \overline{ABD}.$$
$$\overline{ABD}, \overline{AC}D, \overline{B}C\overline{D} \vdash \overline{ABC}.$$

In this theory one of the axioms $\overline{AC}D$ or $\overline{B}C\overline{D}$ is passive. Removing the first of these axioms we have

$$\overline{ABC}, \overline{B}C\overline{D} \vdash \overline{ABD}.$$

Now we have the superfluous literal $\overline{B}$ in the axiom $\overline{ABC}$ or $\overline{B}C\overline{D}$. After removing $\overline{B}$ in one of these axioms we have within automorphism non-trivial theory 1 (see below). □

The other 4 theories are non-trivial:

1. $AB, \overline{B}CD \vdash ACD.$
2. $AB, \overline{ACD}, \overline{B}CD \vdash ACD, B\overline{CD}.$ $(T4')$
3. $A\overline{B}, BC, BD \vdash AC, AD.$
4. $A\overline{D}, B\overline{C}, CD \vdash AB, AC, BD.$

Theory 1 is subtheory of 2, and theory 3 becomes subtheory of 4 after substitution $\{B/D, D/B\}$.

Definition 1.26. The theory is *non-trivial complete* if it is not subtheory of any other theory.

In the given case we have two non-trivial complete theories 2 and 4.

The analogous results we have at $n = 5$. In this case we have 501 theories with passive axioms (see the appendix) and 41 non-trivial theories:

1. $ABC, \overline{A}DE \vdash BCDE.$
2. $ABCE, A\overline{BCE}, BCD\overline{E}, \overline{BCDE} \vdash ABCD, A\overline{BCD}.$
3. $ABC\overline{D}, ABDE, \overline{ABC}\overline{E}, \overline{ABC}D \vdash ABCE, \overline{ABD}\overline{E}.$
4. $ABCE, ABD\overline{E}, \overline{ABCE}, \overline{ABDE} \vdash ABCD, \overline{ABCD}.$
5. $AB, \overline{B}CDE \vdash ACDE.$
6. $ABC, A\overline{CD}E, \overline{A}B\overline{C}D, \overline{BC}D\overline{E} \vdash AB\overline{D}E, \overline{AC}D\overline{E}.$
7. $ABC, A\overline{CD}E, \overline{AB}DE, \overline{BC}D\overline{E} \vdash AB\overline{D}E, \overline{ABC}D.$
8. $ABC, A\overline{B}DE, \overline{A}B\overline{C}E, \overline{AC}D\overline{E} \vdash ACDE, \overline{A}B\overline{C}D.$
9. $ABC, A\overline{B}DE, \overline{A}B\overline{C}D, \overline{ACDE} \vdash ACDE, \overline{A}B\overline{CE}.$
10. $ABC, A\overline{B}DE, \overline{A}B\overline{CD}, \overline{AC}D\overline{E} \vdash ACDE, \overline{A}B\overline{CE}.$
11. $ABD, ACE, \overline{A}B\overline{CE}, \overline{B}C\overline{DE} \vdash A\overline{B}C\overline{D}, B\overline{C}D\overline{E}.$
12. $ABD, ACE, \overline{A}B\overline{CE}, \overline{ABC}\overline{D} \vdash B\overline{C}D\overline{E}, \overline{B}C\overline{D}E.$
13. $AB, \overline{B}CDE, \overline{BCDE} \vdash ACDE, A\overline{CDE}.$

14. $AB, \overline{A}C\overline{DE}, \overline{B}CDE \vdash ACDE, BC\overline{DE}$.
15. $AB, \overline{ACDE}, \overline{B}CDE \vdash ACDE, B\overline{CDE}$.
16. $ABD, AC\overline{D}, \overline{ABC}E, \overline{AB}D\overline{E} \vdash ABC, \overline{ABC}D$.
17. $ABC, \overline{ABD}, \overline{CDE} \vdash AB\overline{D}E, \overline{ABC}E$.
18. $ABD, \overline{A}CE, A\overline{BCD}, \overline{ABCE} \vdash BCDE, \overline{BCDE}$.
19. $ABC, \overline{ABD}, A\overline{BD}E, \overline{A}B\overline{C}E \vdash AC\overline{D}E, \overline{AC}DE$.
20. $ABC, \overline{ABD}, A\overline{BD}E, \overline{A}B\overline{CE} \vdash AC\overline{D}E, \overline{AC}D\overline{E}$.
21. $ABD, \overline{AB}C, A\overline{CDE}, \overline{A}B\overline{DE} \vdash AB\overline{C}E, \overline{A}C\overline{DE}$. $(T5')$
22. $AB, \overline{B}CD, \overline{B}CE \vdash ACD, ACE$.
23. $AB, \overline{B}CD, \overline{BCD}E \vdash ACD, A\overline{CD}E$.
24. $AB, \overline{B}CD, \overline{ACD}E \vdash ACD, B\overline{CD}E$.
25. $ABD, ABE, AC\overline{D}, AC\overline{E}, \overline{ABCE}, \overline{AB}DE \vdash ABC, \overline{ABC}D$.
26. $A\overline{B}C, ADE, \overline{A}B\overline{C}, B\overline{DE} \vdash AC\overline{DE}, B\overline{C}DE$.
27. $ABC, ADE, \overline{ABC}, \overline{ADE} \vdash BC\overline{DE}, \overline{BC}DE$.
28. $AB, ACE, CD\overline{E}, \overline{ACDE} \vdash ACD, B\overline{CDE}$.
29. $AB, CDE, \overline{ABCD}, \overline{BCDE} \vdash A\overline{C}D\overline{E}, \overline{AB}CE$.
30. $ABD, ABE, AC\overline{D}, BC\overline{E}, \overline{ABCE}, \overline{AB}DE \vdash ABC, \overline{ABC}D$.
31. $AB, AC, \overline{B}DE, \overline{C}DE \vdash ADE$.
32. $AB, AC, \overline{A}DE \vdash BDE, CDE$.
33. $AB, \overline{B}CD, \overline{B}CE, \overline{B}DE \vdash ACD, ACE, ADE$.
34. $AB, \overline{B}CD, \overline{B}CE, \overline{ACDE} \vdash ACD, ACE, B\overline{CDE}$.
35. $AC, BD, A\overline{D}E, B\overline{C}E \vdash ABE$.
36. $AC, B\overline{C}, \overline{C}DE \vdash AB, ADE$.
37. $AB, \overline{B}CD, \overline{B}CE, \overline{B}DE, \overline{ACDE} \vdash ACD, ACE, ADE, B\overline{CDE}$.
38. $AC, AD, AE, B\overline{C}, B\overline{D}, B\overline{E} \vdash AB$.
39. $AD, AE, B\overline{D}, C\overline{E} \vdash AB, AC$.
40. $A\overline{B}, BC, BD, BE \vdash AC, AD, AE$.
41. $AD, AE, BD, B\overline{E}, C\overline{D} \vdash AB, AC, BC$.

Subsequent calculation at $n \geqslant 6$ becomes extensive.

We put the result of calculation at $n = 5$ in the next table:

The number of atoms	Theories in all	Non-automorfic	Including					
			with equivalences	with subsumed theorems	with superfluous axioms	with superfluous literals	with-passive axioms	non-trivial theories
1	2	1			1			
2	14	4	1		3			
3	254	20	9		10			1
4	65534	400	275	19	89	6	6	5
5	4294967294	1238099	1158839	66013	12078	616	507	46

The values in this table include degenerated theories too. We can remove these theories if we differ the values in the previous line from the values in the given line (except values in the second column).

Till now we have used consistent theories.

Inconsistent theories become empty at closure. But these theories are very interesting, too.

Let a theory have axioms including:

- all combinations of m positive literals n at a time;
- all combinations of m negative literals n at a time.

So the theory has $2C_m^n$ axioms in total.

If $n \leqslant m < 2n - 1$ then the theory has no theorem. If $m = 2n - 1$ then the theory is complete inconsistent. If $m > 2n - 1$ then the theory is incomplete but joins several complete inconsistent theories.

Definition 1.27. The inconsistent theory is *complete* if this theory becomes consistent at removing an axiom.

The given theory is very interesting. If the theory is complete ($m = 2n - 1$) then a resolvent of any two axioms has $2n - 1$ literals. So initial resolvents are very great if n is great. But a number of literals in resolvents tends to 0 at the next resolutions.

Definition 1.28. The *resolution* is an inference rule to deduce a new axiom (resolvent) from two given ones:

- the resolvent includes all literals of the given axioms except the literals of the contrary pair (the contrary pair is literals differing by signs, one literal from the first axiom, the other is from the second axiom);
- the contrary pair must be unique in the given axioms.

There exist a lot of other complete inconsistent theories. In particular, the set of all perfect clauses ($i = m$, see above) is a complete inconsistent theory.

All this points out incompleteness of theory calculus. There are calculi included (in order of decreasing):

- all theories, both inconsistent and consistent;
- complete inconsistent and consistent theories;
- only consistent theories;
- all non-automorphic theories (without equivalence and subsumed theorems);
- all theories without superfluous axioms and literals;
- all non-trivial theories;
- all complete non-trivial theories.

This list can be continued and extended.

Every restriction from above is realized by additional axioms of the calculus. It means propositional logic is incomplete.

2. Basic Axioms of Propositional Logic.

As we have mentioned, computer-oriented logic is the calculi of theories and axioms.

The theory calculus was investigated in the previous section. In this section we must investigate the axiom calculus.

The axiom calculus is the first order theory. Objects of this theory are the zero order theory.

Below axioms of the zero order theory we shall call just axioms. Axioms of the first order theory we shall call axioms of the calculus. The axiom calculus we shall call just the calculus.

We can formulate four basic problems of the calculus:

- discovery of inconsistency or consistency for a theory;
- closure of a theory (construction of all theorems of the theory);
- construction of all independent axiom systems of a theory;
- reduction to the normal form.

For that we must formulate the axiom calculus.

We use common notation of meta-logic (i.e. the calculus). Logical symbols of negation, conjunction, disjunction, usual and back implication, equivalence, and existence and universal quantifiers:

$$\neg, \wedge, \vee, \rightarrow, \leftarrow, \rightleftarrows, \exists, \forall$$

listed here in order of their priorities.

Below we present axioms of the calculus by rules of classic logic. Other axioms we present in computer-oriented form. In this form we use natural numbers as codes.

Therefore the calculus has the sort N of natural numbers and includes 4 axioms of arithmetic:

$$N_1' \neq N_2' \vee N_1 = N_2 \qquad (Ar1)$$

$$0 = X_1 \rightleftarrows \forall X_2\ X_2' \neq X_1. \qquad (Ar2)$$

$$N_1 \leqslant N_1' \qquad (Ar3)$$

$$N_1' \leqslant N_2 \vee N_2 \leqslant N_1 \qquad (Ar4)$$

The first axiom of arithmetic states the successor function is injective.

The second axiom is 0 definition.

The third axiom introduces the order on the set of the natural numbers.

The fourth axiom $Ar4$ removes all the natural numbers between N_1 and N_1': the number N_2 should be more or equal to N_1' or less than or equal to N_1.

As it follows from two last axioms and from transitivity of an order, the order is linear:

$$N_1 \leqslant N_2 \vee N_2 \leqslant N_1.$$

In more detail about the axioms of arithmetic see [1].

So undefined relations of the calculus are sort N interpreted as the natural number set, one-place function of succession "$\,'\,$" interpreted as addition of 1, and 4-place relation T interpreted as a set of zero order theories and their axioms. Variables of sort N is denoted by $N_0, N_1, N_2, \dots$.

The next axioms of the calculus set property relation T.

This relation for every theory (the first place in T) and for every axiom (the second place) shows atoms (atomic formulas) included in the axioms (the third place) and signs of these atoms (the fourth place).

The sign of the atom is positive, if this atom has no negation, and negative, if this atom has a negation.

The relation T has the kind $N \times N \times N \times b$, where b is the sort interpreted by set {0,1}.

It means theories, axioms, atoms and their signs are coded in T by the natural numbers.

The sort b is a defined notion:

$$b(N_1,) \rightleftarrows N_1 = 0 \vee N_1 = 1 \qquad (D1)$$

In this definition we use anonymous variable (see [1]). Variables of the sort b will be denoted by b_0, b_1, b_2,... .

Example 2.1. If we have axioms AC, $B\overline{C}$ then relation T is a table:

Theory	Axiom	Atom	Atom sign
Ex2.1	0	A	
		C	
	1	B	
		C	¬

In this table we have removed repeated values (in the left part of the table) and have not filled the column "Atom sign" for positive atoms. Besides, natural codes are replaced by usual notation.

This theory has one theorem AB and is a unique non-trivial theory within automorphism on a set of three atoms, more exactly, on a set of six literals. □

This example shows relation T uses computer-oriented presentation of propositional logic formulas without parentheses and logical symbols. Every literal in axiom is met only one time because any relation has not the same tuples. But the axiom system can be inconsistent, i.e. corresponding restrictions are absent.

Now we must state the calculus axioms. These axioms sets the property of the relation T.

First let us replace sorts of axioms and atoms by the types.

The *type* in computational logic is a copy of a sort. The types are denoted by a new symbol differing from a symbol of sort. The types become new sorts with own variables and terms.

We denote type for sort of axioms by A, type for sort of atoms by a. Variables of these types are denoted a_0, a_1, a_2,...,A_0, A_1, A_2,... .

The first calculus axiom sets theory to be numbered in succession from 0:

$$\neg T(N_1',,,) \vee T(N_1,,,) \qquad (A1)$$

In this formula absent places are anonymous. It means these places are filled by variables met in the formula only one time. So we need not give any denotations for these variables. Absent places can be interpreted as removing of these places in n-tuples.

If we number theorems in succession from 0 then codes are assigned to new theories automatically.

Besides, in every theory axioms are numbered in succession from 0, too:

$$\neg T(N_1, A_1',,) \vee T(N_1, A_1,,) \qquad (A2)$$

Similarly atoms are numbered:

$$\neg T(N_1,, a_1',) \vee T(N_1,, a_1,) \qquad (A3)$$

The next two properties of relation T are about subsumed axioms and contrary pairs. Mind that one axiom subsumes another if all literals of the first axiom are in the second axiom. Contrary pairs are the same atoms differing by signs.

A theory has no subsumed axiom:

$$\neg T(N_1, A_1,,) \vee \neg T(N_1, A_2,,) \vee \neg(\forall a_1 \forall b_1\ T(N_1, A_1, a_1, b_1) \to T(N_1, A_2, a_1, b_1)) \vee A_1 = A_2 \qquad (A4)$$

Contrary pairs are absent in the axiom:

$$\neg T(N_1, A_1, a_1, b_1) \vee \neg T(N_1, A_1, a_1, b_2) \vee b_1 = b_2 \qquad (A5)$$

Therefore relation T is a function.

Two theories are equal if they have the same atoms. Equal theories are absent in T:

$$\neg T(N_1,,,) \vee \neg T(N_2,,,) \vee \neg(\forall A_1 \forall a_1 \forall b_1\ T(N_1, A_1, a_1, b_1) \rightleftarrows T(N_2, A_1, a_1, b_1)) \vee N_1 = N_2 \qquad (A6)$$

The next group of calculus axioms settles every theory must be in the normal form (see section 4.3). In this form all theories are the same, if they are automorphic over the literal set. Mind that two theories are automorphic if one of these theories equal the other after a substitution of the literals.

We introduce an order in the theory set to reduce to the normal form. By brief searching through substitutions we find the substitution to give minimal theory over substitution set. Reduction to the minimal theory is reduction to the normal form. In this form all automorphic theories become the same.

So the theory form is normal if every literal substitution does not reduce theory.

Corresponding axioms of the calculus define the order in theory set and the normal form of theorem presentation.

The list of axioms of the calculus defining properties of relation T is finished.

There are other axioms of the calculus defining the resolution, the closed theories, the independent axiom systems and inconsistent theories.

Besides, there are calculus axioms for rules of:

- seeking inconsistent theories,
- constructing closed theories and all independent axiom systems,
- reducing to the normal form,
- seeking equivalencies, subsumed theorems, superfluous axioms, passive axioms and passive literals.

All these rules are valid formulas. The zero order valid formulas can not be used in the zero order theories, but they can be used in the calculus, i.e. in the first order theory.

3. Inconsistent Theories.

The theory is *inconsistent* if the empty theorem is deduced.

The inconsistent theories are used to prove theorems. Negation of a theorem is added to an axiom set of a theory and the theory becomes inconsistent. Proving inconsistency of this joint theory we prove the theorem too.

The given section consists of two parts.

In the first part we give formal definition of resolution rule.

In the second part this rule is used to prove inconsistency of theories.

3.1. Resolution Rule.

The resolution is the basic inference rule in computational logic. This rule uses the relation $Res(N_1, A_1, A_2, A_3)$. This relation sets an axiom A_3 as resolvent (the result of resolution) of axioms A_1 and A_2 in a theory N_1.

At first we define the relation $res(N_1, A_1, A_2, a_1)$. This relation takes axioms A_1 and A_2 with a_1 as a base of a contrary pair. Atom a_1 is in both axioms but with different signs. This atom with two signs is the *contrary pair*. The contrary pair must be unique in axioms A_1 and A_2.

Definition 3.2 of relation res:

$$res(N_1, A_1, A_2, a_1) \rightleftarrows (\exists b_1 \exists b_2\ T(N_1, A_1, a_1, b_1) \wedge T(N_1, A_1, a_1, b_2) \wedge b_1 \neq b_2) \wedge$$
$$(\forall a_2 \forall b_1 \forall b_2\ T(N_1, A_1, a_2, b_1) \wedge T(N_1, A_2, a_2, b_2) \wedge b_1 \neq b_2 \rightarrow a_2 = a_1) \qquad (D2)$$

So a contrary pair must exist and be unique.

Now we can define relation Res.

Definition 3.3 of relation Res:

$$Res(N_1, A_1, A_2, A_3) \rightleftarrows \exists a_1\ res(N_1, A_1, A_2, a_1) \wedge \forall a_2 \forall b_1 T(N_1, A_3, a_2.b_1) \rightleftarrows$$
$$a_2 \neq a_1 \wedge (T(N_1, A_2, a_2, b_1) \vee T(N_1, A_3, a_2, b_1)) \qquad (D3)$$

So axiom A_3 is deduced from axioms A_1 and A_2 in theory N_1, if A_1 and A_2 have a unique contrary pair and A_3 includes all literals of axioms A_1 and A_2 except literals of the contrary pair.

Example 3.4. In theory $(T3')$: $AC, B\overline{C}, AB$, the resolution of two first axioms equals the third axiom. The contrary pair includes C. □

Using resolution rule we can construct a closure of a theory N_1. If this closure exists then theory N_1 is consistent. If this closure does not exist then theory N_1 is inconsistent.

But there exists a more effective way to prove inconsistency. We must reduce a number of atoms in the theory.

3.2. Rule of Atom Removing.

The proof of theory inconsistency is realized in several steps.

Rule 3.5 of sequential removing of atoms:

- negation of a theorem is added to axioms of a theory;
- a maximal atom is chosen;
- resolutions with the maximal atom as a base of a contrary pair are executed;
- subsumed arguments of resolutions are removed (an argument is subsumed if its length is less then the length of the resolvent, the length equals a number of literals);
- information about resolutions is placed in a special relation;
- axioms with the maximal atom are removed;
- 5 previous steps are repeated until atoms exist;
- resolutions reduced to the empty resolvent are extracted from the special relation mentioned above.

The resolutions, reducing to the empty resolvent, form the proof of the given theorem. A number of proofs equals a number of the empty resolvent.

We can choose the proof with a minimal number of steps.

The formalized rule of sequential removing of atoms is present in [3].

Theorem 3.6 *The rule of sequential removing of atoms allows to find every inconsistent theory.*

Proof: We must prove this rule coincides with the rule of disintegration [2].

Let the theory be:

$$A_1a_0, ..., A_ma_0,\ A_{m+1}\overline{a}_0, ..., A_n\overline{a}_0,\ T_0$$

where a_0 - removing atom, $A_1, ..., A_n$ are axiom remainders after removing a_0, T_0 is a part of the theory without a_0.

According to [2] T_1 and T_2

$$T_1 = (A_1, ..., A_m, T_0),\ T_2 = (A_{m+1}, ..., A_n, T_0).$$

are a disintegration of the initial axiom sets. So the disjunction of T_1 and T_2 is inconsistent if the initial axiom set is inconsistent.

We must show this disjunction consists of all resolvents and T_0.

This disjunction equals (we use tradition notation too):

$$(T_1 \vee T_2) = (\ ((A_1, ..., A_m) \vee (A_{m+1}, ..., A_n)) \wedge ((A_1, ..., A_m) \vee T_0) \wedge ((A_{m+1}, ..., A_n) \vee T_0) \wedge (T_0 \vee T_0)\ )$$

From

$(T_0 \vee T_0) = T_0$,

$(A_1, ..., A_m) \vee T_0$ subsumed by T_0,

$(A_{m+1}, ..., A_n) \vee T_0$ subsumed by T_0 too,

we get:

$$(T_1 \vee T_2) = (\quad ((A_1, ..., A_m) \vee (A_{m+1}, ..., A_n)) \wedge T_0 \quad).$$

So disjunction of T_1 and T_2 includes T_0. We must show the other part of this disjunction consists of all resolvents.

But the formula $((A_1, ..., A_m) \vee (A_{m+1}, ..., A_n))$ equals

$$A_1A_{m+1}, ..., A_1A_n,\ A_2A_{m+1}, ..., A_2A_n,\ ...,\ ...,\ A_mA_{m+1}, ..., A_mA_n$$

This formula consists of all normal and abnormal resolvents.

The resolvent is *abnormal* if its arguments have two or more contrary pairs. This resolvent is a valid formula and can be removed. Then the disjunction of T_1 and T_2 consists of T_0 and all normal resolvents. □

Example 3.7. Theory $AB, AC, BC, \overline{AB}, \overline{AC}, \overline{BC}$ is inconsistent. We must prove this.

Setting the order $A < \overline{A} < B < \overline{B} < C < \overline{C}$ we have the next order of axioms:

1. CB (the beginning of axioms with positive C)
2. CA

3. $\overline{CB}$ (the beginning of axioms with negative C)
4. $\overline{CA}$
5. BA (the beginning of axioms without C)
6. $\overline{BA}$

The order allows to remove superfluous combinations of axioms for resolution. Instead of 15 combinations we have only 4: 1 with 3, 1 with 4, 2 with 3, and 2 with 4. In two of these combinations we have resolvents $B\overline{A}$ and $\overline{B}A$.

After removing axioms with C and $\overline{C}$ we have:

1. BA (the beginning of axioms with positive B)
2. $B\overline{A}$
3. $\overline{B}A$ (the beginning of axioms with negative B)
4. $\overline{BA}$

At resolution of axioms 1 and 3 we have resolvent A. This resolvent subsumed both axioms 1 and 3. So we can omit resolutions 1 with 4 and 2 with 3.

After removing axioms with B and $\overline{B}$ we have:

1. A
2. $\overline{A}$

Resolution 1 with 2 gives the empty resolvent.

This resolvent is unique. So the proof is unique too.

The total number of used combinations equals 7. This number defines the time of iteration execution.

It is known a number K_s of combinations to order a set of axioms equals $nN\log_2 N$, where n equals a number of atoms (n=3) and N equals a number of axioms (N=6). Below we show K_s to be less then a number of iterations.

Example 3.4 is a particular case of theories over $2n-1$ atoms and with every possible positive or negative axioms of length n. It means positive (negative) axioms consist only of positive (negative) literals.

In this example n=2. For other n we have the next table:

n	L	K	K_s/K	k
1	2	1	2.00	
2	12	7	4.43	1.08
3	60	79	3.28	1.50
4	280	1011	1.70	1.65
5	1260	13231	0.76	1.71
6	5544	176583	0.31	1.75
7	24024	2408183	0.12	1.78
8	102960	33494947	0.04	1.81
9	437580	473585695	0.01	1.83
10	1847560	6787041571	0.00	1.85
∞			0.00	2.00

In this table n is the length of every axiom, L is the total length of all initial axioms, K is a number of iteration combinations, K_s/K is the ratio of a number of order combinations to a number of iteration combination, $k = \lg(K_n/K_{n-1})/\lg(L_n/L_{n-1})$ is exponent for time dependence, K_n and K_{n-1} are values of K in line n and in the previous line, L_n and L_{n-1} are the values of L in the lines n and $n-1$.

As follows from this table, the time dependence is quadratic at large n. □

Example 3.8. Complete perfect CNF is inconsistent.

In this example resolvents subsume initial axioms. On every step axioms have the same length and resolvent length is less then axiom length on 1.

In this case time dependence is linear at great n.

The results are in the next table:

n	L	K	k
1	2	1	
2	8	3	0.79
3	24	7	0.77
4	64	15	0.78
5	160	31	0.79
6	384	63	0.81
7	896	127	0.83
8	2048	255	0.84
9	4608	511	0.85
10	10240	1023	0.87
∞			1.00

□

4. Theory Closure.

This section has 4 parts.

In the first part we give high effective rules for construction of closed theories.

In the second part we use closed theories for construction of all independent axiom systems of a theory. The rule for this construction is highly effective too.

In the third part we use closed theories for reduction to the normal form. In this form all automorphic theories become the same.

In the last part we give rules to find equivalencies, superfluous and passive axioms and passive literals.

4.1. Rule of Level Removing.

A theory is *closed* if it contains all theorems except subsumed:

$$C(N_1) \rightleftarrows \forall A_1 \forall A_2 \exists A_3\ \neg T(N_1, A_1,,) \vee \neg T(N_1, A_2,,) \vee (T(N_1, A_3,,) \wedge Res(A_1, A_2, A_3))$$

where C is the relation "to be the closed theory".

If a theory is inconsistent then the closure is empty. In this case we must use the rule of sequential removing of atoms. In other cases this rule is not applied to theory closure.

Example 4.1. Let a theory have axioms AB, $\overline{A}C$, and $\overline{B}D$. Using the rule of sequential removing of atoms we have no new axioms. So we lost theorems AD and BC (after removing atoms D and C). □

Therefore we must use other rules.

Rule 4.2 of sequential removing of axioms:

- an axiom is removed after resolutions with every other axioms.

It means every axiom does not give new theorems after resolutions with other axioms.

Theorem 4.3. *The rule of sequential removing of axioms is a valid formula.*

Proof: At proving we use three properties of resolution (we use symbol " $\circ$ " to note the resolution of two axioms):

- resolution is commutative $A_1 \circ A_2 = A_2 \circ A_1$;
- resolution of the same axioms $A_1 \circ A_1$ does not exist;
- repeated resolution with the same axiom $A_1 \circ (A_1 \circ A_2)$ does not exist.

Let A_0 be a removing axiom, $A_1, ..., A_m$ be other axioms of a theory.

After removing A_0 we have axioms $A_1, ..., A_m, A_0 \circ A_1, ...A_0 \circ A_m$.

We must show resolution A_0 with any axioms appearing at further iteration not to generate a new axioms.

Before iteration it is clear: resolution A_0 with the axioms $A_1, ..., A_m, A_0 \circ A_1, ...A_0 \circ A_m$ does not generate new axioms.

Let the theorem be deduced till iteration step n inclusive. Let us show this theorem is deduced at step $n+1$, too.

Let A_i and A_j be arbitrary axioms appearing at step n, $A_i \circ A_j$ be an axiom appearing at step $n+1$, a_0 be an atom of a contrary pair for resolution A_0 and $A_i \circ A_j$ ($a_0 \in A_0$), a_{ij} be an atom of a contrary pair for resolution A_i and A_j ($a_{ij} \in A_i$), $\overline{a}_{ij} \in A_j$, possible $a_{ij} \in A_0$, but always $\overline{a}_{ij} \notin A_0$).

We shall show this resolution does not generate new axioms.

The table below has all the cases of resolution of A_0 and $A_i \circ A_j$. In this table $A_k < A_l$ means axiom A_l to subsume axiom A_k ($a_0 \in A_0, \overline{a}_{ij} \notin A_0$):

1	$a_{ij} \notin A_0$	$\overline{a}_0 \in A_i$	$a_{ij} \in A_i$	$\overline{a}_0 \in A_j$	$a_{ij} \in A_j$	$A_0 \circ (A_i \circ A_j) = (A_0 \circ A_i) \circ (A_0 \circ A_j)$
2		$\overline{a}_0 \in A_i$	$a_{ij} \in A_i$	$\overline{a}_0 \notin A_j$	$a_{ij} \in A_j$	$A_0 \circ (A_i \circ A_j) = (A_0 \circ A_i) \circ A_j$
3		$\overline{a}_0 \notin A_i$	$a_{ij} \in A_i$	$\overline{a}_0 \in A_j$	$a_{ij} \in A_j$	$A_0 \circ (A_i \circ A_j) = (A_0 \circ A_j) \circ A_i$
4	$a_{ij} \in A_0$	$\overline{a}_0 \in A_i$	$a_{ij} \in A_i$	$\overline{a}_0 \in A_j$	$a_{ij} \in A_j$	$A_0 \circ (A_i \circ A_j) < A_0 \circ A_i$
5		$\overline{a}_0 \in A_i$	$a_{ij} \in A_i$	$\overline{a}_0 \notin A_j$	$a_{ij} \in A_j$	$A_0 \circ (A_i \circ A_j) < A_0 \circ A_i$
6		$\overline{a}_0 \notin A_i$	$a_{ij} \in A_i$	$\overline{a}_0 \in A_j$	$a_{ij} \in A_j$	$A_0 \circ (A_i \circ A_j) < A_i$

As follows from this table the resolvent $A_0 \circ (A_i \circ A_j)$. is not new in generation of A_0. □

The rule of sequential removing of axioms is not high effective but it is used in a more effective rule of level removing.

Definition 4.4.:

- *level 0* is an initial axiom set (as a rule this set has not theorems);

- *level* $k + 1$ is the set of the axioms generated at resolutions of all possible pairs from axioms of level k.

Definition 4.5. The *generalized theorem* is a resolvent with members consisting of:

- the number 0 and atom from a contrary pair;
- the number 1 and literals met in an axiom with a positive literal of a contrary pair;
- the number 2 and literals met in axiom with negative literal of a contrary pair;
- the number 3 and literals met in both axioms.

So the generalized theorem has all its literals (marked by the numbers 1, 2 and 3) and the information about resolution generating this theorem.

Rule 4.6 of level removing (for generalized theorems):

- after constructing level $k + 1$ we can remove level k.

It means we use only theorems of level $k + 1$ in the next step of iteration.

Theorem 4.7. *The rule of level removing constructs all theorems of a theory.*

Proof: We can limit ourselves by proving of level 0 deletion.

If level 0 has one axiom the theorem is obvious.

Let the theorem be true for level 0 with n axioms: $A_1, .., A_n$. We shall show this theorem is true for level 0 with one more axiom A_{n+1}.

We must show axiom A_{n+1} can be removed after resolutions with all other axioms of level 0.

All resolvents $A_{n+1} \circ A_i$ ($i \leqslant n$) belong to level 1 according to the level definition.

We shall show all resolvents $A_{n+1} \circ (A_i \circ A_j)$ ($i \leqslant n, j \leqslant n$) belong to level 1 or are generated by level 1.

Replacing A_0 by A_{n+1}) in the table above we see resolvent $A_{n+1} \circ (A_i \circ A_j)$:

- equals resolvent $(A_{n+1} \circ A_i) \circ (A_i \circ A_j)$ (line 1 of the table);
- is subsumed by resolvent $A_{n+1} \circ A_i$ (lines 4 and 5);
- is subsumed by axiom A_i (line 6).

For other two cases we use the information in generalized theorems of level 1. This information lets extract A_j from $A_i \circ A_j$ and construct resolvent $(A_{n+1} \circ A_i) \circ A_j$ in the first case (line 2). In the second case (line 3) we extract A_i from $A_i \circ A_j$ and construct resolvent $(A_{n+1} \circ A_j) \circ A_i$. In both cases constructed resolvents equal $A_{n+1} \circ (A_i \circ A_j)$.

Therefore we can construct resolution of axioms for the next level 2. One of these axioms belongs level 1 and the other belongs to level 0 though this level is absent. All necessary information about level 0 is taken from generalized theorems of level 1. So all theorems generated by A_{n+1} belong to level 1 and 2.

It means axiom A_{n+1} can be removed from iteration process after construction of level 1. All theorems deduced from this axiom will be constructed without its participation.

So the proving theorem is deduced by induction. □

As follow from this theorem the resolution of every level axioms must be united with resolutions of axioms for all previous levels only by using information in generalized theorems.

Level removing rule is high effective because extra resolutions of two theorems are constructed in the presence of the conditions:

- usual resolvent of these theorems does not exist;

- the theorems have common father, i.e. common axiom of previous level;

- an atom of a contrary pair in one theorem does not equal an atom of a contrary pair in another theorem (otherwise resolvent of one of these theorems and the previous level axiom extracted from the other theorem does not exist);

- from other two axioms extracted from the theorems we take axioms with greater atom of a contrary pair.

If these conditions take place we construct extra resolution of a taken axiom from one theorem with the other theorem.

Example 4.8 Let five axioms be given at level 0:

1. $ABCDE$.
2. $\overline{A}F$.
3. $\overline{B}G$.
4. $\overline{C}H$.
5. $\overline{D}I$.

We must construct level 1 (in parentheses we give the serial numbers of axioms taking part in resolutions):

6. (1,2) $A^0B^1C^1D^1E^1F^2$.
7. (1,3) $A^1B^0C^1D^1E^1G^2$.
8. (1,4) $A^1B^1C^0D^1E^1H^2$.
9. (1,5) $A^1B^1C^1D^1E^0I^2$.

Now we construct level 2 (in parentheses after the point we indicate the serial number of axiom extracted from generalized theorem).

10. (6,7.2) $B^0C^1D^1E^1F^1G^2$.
11. (6,8.2) $B^1C^0D^1E^1F^1H^2$.
12. (6,9.2) $B^1C^1D^0E^1F^1I^2$.
13. (7,8.2) $A^1C^0D^1E^1G^1H^2$.
14. (7,9.2) $A^1C^1D^0E^1G^1I^2$.
15. (8,9.2) $A^1B^1D^0E^1H^1I^2$.

Level 3 is constructed:

16. (10,11.2) $C^0D^1E^1F^1G^1H^2$.
17. (10,12.2) $C^1D^0E^1F^1G^1I^2$.
18. (11,12.2) $B^1D^0E^1F^1H^1I^2$.
19. (13,14.2) $A^1D^0E^1G^1H^1I^2$.

Level 4 is:

20. (16,17.2) $D^0E^1F^1G^1H^1I^2$.

It is the end of level iteration. The closed theory includes all axioms of all levels. We have total 20 axioms. □

4.2. Rule of Independent Axiom System Construction.

Below we exclude theories that have subsumed theorems. For that we construct again all resolvents.

Rule 4.9 of resolvent construction:

- for every pair of axioms of a closed theory we construct the resolvent;

- this resolvent is looked for among other axioms;

- the resolvent is treated as subsumed if it is not found, in that case construction of independent axiom systems is blocked;

- if finding the resolvent we create the triple of three serial numbers of axioms, two of them are initial for resolution and one more is their resolvent;

- simultaneously we fill the list of resolvents and the list of resolution arguments;

- if a number of resolutions (triples) equals 0, then the theory has no theorem and construction of the independent axiom systems is blocked;

- if the list of resolvents does not contain all axioms then there are unused axioms and construction of independent axiom systems is blocked too; else

- we fill another list, the list of axioms not to be resolvents (these axioms are independent).

Rule 4.10 of theory construction with the unique independent axiom system:

- the total axiom list before iteration is a list of axioms not to be resolvents, iteration begins with examination of a resolution set (every set member is a triple consisting of three numbers, the first two to be the serial numbers of initial axioms and the third to be the serial number of resolvents);

- if the third number of a triple is absent in the total axiom list and the first two numbers are present in this list then the third number is added to the list;

- if after examination of all resolutions (triples) the total axiom list is changed, then we examine the resolution set again; else

- if the total axiom list contains all axioms of the theory then the theory has unique independent axiom system, this system consists of axioms not to be resolvent.

If a total axiom list contains only part of theory axioms, then we must use one more rule.

Rule 4.11 of theory construction with several independent axiom systems:

- before iteration list of the independent axioms equals list axioms not to be resolvents, a total axiom list equals the axiom list from the previous rule, a state list and set M (see below) are empty, iteration begins with the examination of the resolution set;

- if the third number of a resolution set member is absent in the total axiom list but the first two numbers are presented in this list then the third number is added to the total axiom list;

- if the third number is absent in the total axiom list and one or both of the other numbers are absent in this list too, then state information is included in state list (state information consists of the third number, the total axiom list, and the independent axiom list), the third number is added to the total axiom list, and the other numbers, absented in the total axiom list, is added to this list and to the independent axiom list too;

- if after examining of all resolution set we have the total axiom list to complete then the independent axiom list is put in set M as its member and in any case the last state from the state list is restored, the restored third number is increased by 1, the restored state is removed from the state list, and iteration is continued from a new state;

- if we must restore the last state but the state list is empty then iteration is finished.

After iteration is finished set M contains all independent axiom systems of the given theory.

We must remove from set M all systems subsumed by other systems. We must remove automorphic systems, too.

This rule is high effective because we use only true combination of triples. If a theory has a unique system then the combination of triples is absent.

Example 4.12. Let axiom of the closed theory be numerated:

$$1: ABC,\ \ 2: ABD,\ \ 3: AC\overline{D},\ \ 4: \overline{ABC},\ \ 5: \overline{ABD},\ \ 6: \overline{AC}D,\ \ 7: B\overline{C}D,\ \ 8: \overline{B}C\overline{D}$$

This theory has 8 resolutions with triples (the first two are the serial numbers of initial axioms, the third is the serial number of resolvent):

$1:(1,7,2)$. $2:(1,8,3)$. $3:(2,3,1)$. $4:(2,6,7)$. $5:(3,5,8)$. $6:(4,7,6)$. $7:(4,8,5)$. $8:(5,6,4)$.

In the table below we present steps of iteration to build independent axiom systems. Before iteration the lists of independent and dependent axioms are empty. The list of dependent axioms (theorems) is the total axiom list without independent axioms.

We use the following notation.

In column 3 we add the symbol "-" to the state number if this state will be restored at the end of the resolution list. We add the symbol "–" in this column if this state will be restored at once after execution of this step (because a new independent axiom system is completed) and does not change till the end of the resolution list. In the first case a new state equals an old one, in the second case the number of a new state is less then the number of an old state for 1.

In column 7 the independent axiom list is marked by "+" or "−" if this list is completed. In the first case the list is new, in the second case the list is old, met above.

As follows from the example we combine resolutions (triples) instead of axioms.

On steps 1-13 we combine resolution (triple) 1 with other resolutions. On steps 14-20 we combine resolution 2 with other resolutions except resolution 1, and so on.

On step 1 we convince the initial axioms for resolution 1 not to form an independent system.

On step 2 we convince the initial axioms for resolution 1 and 2 not to form an independent system too.

Step	Re-storing	Re-membering	Triple	Before step execution		After step execution	
				Independent axioms	Theorems	Independent axioms	Theorems
1	2	3	4	5	6	7	8
1		1	1			1,7	2
2		2	2	1,7	2	1,7,8	2,3
3		3	6	1,7,8	2,3	1,4,7,8	2,3,6
4			7	1,4,7,8	2,3,6	1,4,7,8+	2,3,5,6
5	3	3	7	1,7,8	2,3	1,4,7,8−	2,3,5
6	3	3–	8	1,7,8	2,3	1,5,6,7,8+	2,3,4
7	2	2	5	1,7	2	1,3,5,7	2,8
8		3	6	1,3,5,7	2,8	1,3,4,5,7+	2,6,8
9	3	3–	8	1,3,5,7	2,8	1,3,5,6,7+	2,4,8
10	2	2	6	1,7	2	1,4,7	2,6
11		3–	7	1,4,7	2,6	1,4,7,8−	2,5,6
12	2	2	7	1,7	2	1,4,7,8−	2,5
13	2	2–	8	1,7	2	1,5,6,7	2,4
14	1	1	2			1,8	3
15		2	4	1,8	3	1,2,6,8	3,7
16		3	7	1,2,6,8	3,7	1,2,4,6,8+	3,5,7
17	3	3–	8	1,2,6,8	3,7	1,2,5,6,8+	3,4,7
18	2	2	6	1,8	3	1,4,7,8−	3,6
19	2	2-	7	1,8	3	1,4,8	3,5
20	2	2–	8	1,8	3	1,5,6,8	3,4
21	1	1	3			2,3	1
22		2	4	2,3	1	2,3,6	1,7
23		3	5	2,3,6	1,7	2,3,5,6	1,7,8
24			8	2,3,5,6	1,7,8	2,3,5,6+	1,4,7,8
25	3	3	7	2,3,6	1,7	2,3,4,6,8+	1,7,5
26	3	3–	8	2,3,6	1,7	2,3,5,6−	1,4,7
27	2	2	5	2,3	1	2,3,5	1,8
28		3	6	2,3,5	1,8	2,3,4,5,7+	1,6,8
29	3	3–	8	2,3,5	1,8	2,3,5,6−	1,4,8
30	2	2	6	2,3	1	2,3,4,7	1,6
31		3–	7	2,3,4,7	1,6	2,3,4,7,8+	1,5,6
32	2	2-	7	2,3	1	2,3,4,8	1,5
33	2	2–	8	2,3	1	2,3,5,6	1,4
34	1	1	4			2,6	7
35		2	5	2,6	7	2,3,5,6−	7,8
36	2	2-	7	2,6	7	2,4,8,6	5,7
37	2	2–	8	2,6	7	2,5,6	4,7
38	1	1	5			3,5	8
39		2–	6	3,5	8	3,4,5,7	6,8
40	1	1	6			4,7	6
41		2–	7	4,7	6	4,7,8	5,6
42	1	1-	7			4,8	5
43	1	1–	8			5,6	4

On step 3 we see the initial axioms for resolution 1, 2 and 3 to terminate further combinations (1, 2, 3 and 4, 1, 2, 3 and 5 etc). The same result we have for resolutions 1, 2 and 4, 1, 2 and 5. But initial axioms for resolutions 1, 2 and 6 do not terminate further combinations.

On step 4 initial axioms for resolutions 1, 2, 6 and 7 form a completed independent system.

Therefore every combination including resolutions 1, 2, 6 and 7 can be ignored. Combinations including resolutions of 1, 2 and 6 can be ignored too because of step 3. So we use combination 1, 2 and 7 at the next step.

On step 5 the initial axioms for resolution 1, 2 and 7 form a complete independent system, but this system is old. The next combination includes resolutions 1, 2 and 8.

On step 6 the initial axioms for resolution 1, 2 and 8 form a complete independent system and this system is new. The next combination includes resolutions 1 and 3.

On step 7 we see the combination of resolutions 1 and 3, 1 and 4 to be ignored. The next combination includes resolutions 1 and 5.

On step 8 the initial axioms of resolutions 1, 5 and 6 form a new complete independent system. The next combination of resolutions is 1, 5, and 7.

On step 9 a new complete independent system is formed by axioms of resolutions 1, 5 and 8. Axioms of resolutions 1, 5, and 7 were ignored.

On step 10 we examine resolution 1 and 6, on step 11 we have an old complete independent system for resolutions 1, 6, and 7.

On step 12 we remove the combination of resolutions 1, 6 and 8, but we have an old complete independent system for resolutions 1 and 7.

We remove the combination of resolutions 1 and 8 as it follows from step 13 and we finish to examine combinations including resolution 1. Such combinations are 128 but we used only 13 steps.

The next steps use the other combinations.

The given example demonstrates effectiveness of the rule above. In this rule total combinations of axioms is replaced by less account of resolution combinations. □

4.3. Rules of Reduction to Normal Form.

The simplest way removing automorphic theories is reduction to form such that all automorphic theories become the same. This form is normal for theories of order 0 because it lets us have an unique representation for all automorphic theories.

Reduction to a normal form is possible for any theory. But the greatest effect we have at reduction of closed theories and independent axiom systems. In other cases the normal form of a theory is dependent on theorems included in this theory.

Rule 4.13 basic:

- all axioms are grouped according to theirs length (axiom length equals a number of literals in the axiom);
- the groups are ordered by the increase of length;
- in every group for every atom we calculate the number of occurrences of this atom n_+ as a positive atom and the number of occurrences n_- as a negative atom;
- we change this atom sign by opposite in every occurrence if $n_+ < n_-$ in the group or we do not change the atom sign if $n_+ > n_-$, in both cases group iteration is finished for this atom;
- we continue group iteration if $n_+ = n_-$;
- we finish group iteration for this atom (if $n_+ = n_-$ in all groups the atom signs is not changed) and begin new iteration for the next atom;
- we finish all iterations;
- we use the rule of partial descending ordering of atoms by value of $n_+ + n_-$ or n_+ if sums are equal (see below all the rules);
- we use the rule of partial ascending ordering of axioms by value of length or by value of literals if lengths are equal;
- we use the rule full ordering of atoms and axioms;
- we use the new notation (new codes) for ordered atoms.

The reduction to normal form is finished.

The next rule is recursive. The initial data are changed at every recursion.

Rule 4.14 of partial atom descending ordering.

If we have:

- a group of axioms (with the same length of axioms),
- a list of atoms (necessary to be divided in subgroups with the same frequency characteristics),
- frequency characteristics of atoms (i.e. the numbers of occurrences of atoms n_+ and n_- as positive and negative literals) for each group axioms,

then we:

- go to the next group of axioms (with a large length), if all atoms have zero frequencies;

- create a subgroup including all atoms of the list if all atoms have zero frequencies in the last group;

- order the atom list by value of $n_+ + n_-$ or n_+ if sums are equal, in both cases frequencies must be non-zero;

- create atom sublists with the same frequency characteristics (for further ordering using frequency characteristics of the next axiom groups);

- create an atom subgroup if a sublist has unique atom;

- execute this rule one more for every multiatom sublist if a given axiom group is not the last one;

- create an atom subgroup for every atom sublist if a given axiom group is the last one.

So every atom sublist is divided into several atom sublists if an axiom group is not last. If the axiom group is the last then every sublist becomes final and we rename it as an atom subgroup.

As a result of the rule execution we have atom subgroups. The set of subgroups is ordered by frequency characteristics but atoms into subgroups are disordered because these atoms have the same frequency characteristics in every axiom group.

Among atom subgroups we take ones with $n_+ = n_-$ for all axiom groups. Atoms of these subgroups can change signs (other atoms have final signs, see above). In the next rule we set an order in theories and arbitrarily change atom signs. If the changed theory is less than initial one then the changed theory replaces the initial one.

Rule 4.15 of full ordering of atoms and axioms:

- we put an arbitrary atom order into every atom subgroup;

- we put partial axiom order in a theory set (axioms are ordered by theirs length, or by literals at the same length of axioms, literals are ordered by a sign, or by an atom subgroup at the same sign);

- we put iteration of atom subgroups and of atoms into subgroups;

- if an atom subgroup has $n_+ = n_-$ then we create a copy of initial theory for an atom of this subgroup;

- we change the sign of this atom in the theory copy;

- the initial theory is replaced by its copy if the copy is less then the initial theory;

- now we finish to change signs in atom subgroups with $n_+ = n_-$ and below we use any subgroup and any atom in this subgroup;

- we create again a copy of the initial theory;

- in this copy an atom and the next to it (in the atom subgroup) are changed by notation (the code of one of atoms is changed by the code of the other and the code of the other is changed by the code of one);

- the initial theory is replaced by the copy if the copy is less than the initial theory;

- after atom subgroups are finished we repeat seven last steps if an initial theory was replaced at execution of these steps;

As a result of the execution of this rule a theory is minimized and reduces to the normal form.

Example 4.16. Let a theory be represented by one of independent axiom system (from example 4.12):

$$ABC, \overline{ABC}, B\overline{C}D, \overline{B}C\overline{D}. \tag{1}$$

Axioms of this theory have the same length and form the unique axiom group (with the same frequency characteristics).

Atoms of this theory have the next frequency characteristics n_+ and n_-:

$$A: 1, 1. \quad B: 2, 2. \quad C: 2: 2. \quad D: 1, 1$$

For all atoms $n_+ = n_-$. Therefore signs of atoms are not changed.

In accordance with the frequency characteristics we have two atom subgroups: $\{A, D\}$ and $\{B, C\}$. These subgroups can be ordered by decrease of their frequency characteristics: $\{B, C\} > \{A, D\}$. Using this ordering we do next substitutions (in numerator we have a replaced notion, in denominator we have a replacing notion):

A/C, B/A, C/B

As a result we have:

$$CAB, \overline{CAB}, A\overline{B}D, \overline{A}B\overline{D}$$

or

$$ABC, \overline{ABC}, A\overline{B}D, \overline{A}B\overline{D}.$$

Then we do substitutions of axioms in accordance with the order of axioms:

$$ABC, A\overline{B}D, \overline{A}B\overline{D}, \overline{ABC}. \qquad (2)$$

In this form atoms create atom subgroups $\{A, B\}$ and $\{C, D\}$. For both subgroups $n_+ = n_-$.

We construct a copy of (2) and change the signs of literals containing A:

$$\overline{A}BC, \overline{AB}D, AB\overline{D}, A\overline{BC}.$$

After substitutions of axioms we have:

$$AB\overline{D}, A\overline{BC}, \overline{A}BC, \overline{AB}D.$$

Because theory (2) is less than this copy the initial theory is not changed.

Again we construct a copy of (2) and change the signs of literals contained B:

$$A\overline{B}C, ABD, \overline{ABD}, \overline{A}B\overline{C}$$

or

$$ABD, A\overline{B}C, \overline{A}B\overline{C}, \overline{ABD}$$

Again theory (2) is less than this copy.

Now we construct a copy of (2) one more and rename atoms A and B in the copy:

$$BAC, B\overline{A}D, \overline{B}A\overline{D}, \overline{BAC}$$

or

$$ABC, A\overline{BD}, \overline{A}BD, \overline{ABC}$$

This copy does not replace theory (2).

We must go to the next atom subgroup.

Changing the signs of literals containing C we get the copy:

$$AB\overline{C}, A\overline{B}D, \overline{A}B\overline{D}, \overline{AB}C.$$

Substitutions of axioms are unnecessary. The received copy does not replace theory (2).

We change the signs of literals containing D:

$$ABC, A\overline{BD}, \overline{A}BD, \overline{ABC}.$$

Again theory (2) is not replaced.

Now we rename atoms C and D:

$$ABD, A\overline{B}C, \overline{A}B\overline{C}, \overline{ABD}.$$

Theory (2) has no change.

Further transformations are not realized because theory (2) had no change.

The execution of the normalization rule is finished. The normal form of theory (1) is theory (2).

We have analogous transformations of one more independent axiom system of example 4.12:

$$ABD, AC\overline{D}, \overline{ABD}, \overline{AC}D$$

and these transformations lead again to normal form (2). Therefore both independent axiom systems are automorphic.

The other independent axiom systems of example 4.12 create one more normal form. So instead of 10 systems we have only 2 non-automorphic systems. □

4.4. Rules of Theory Perfection.

In section 1 we showed that theories became more perfect if equivalencies, subsumptions, superfluous axioms, and passive axioms and literals are removed.

First we shall remove the equivalencies. Below the symbol of equivalence "$\rightleftarrows$" belongs to propositional logic (i.e. this symbol is the object of theory of order 0).

If a theory has an axiom as $AB...(CD...E \rightleftarrows FG...H)$, then this axiom transforms into two axiom groups at reduction to CNF:

$$AB...CD...E\overline{F}, AB...CD...E\overline{G}, ..., AB...CD...E\overline{H},$$

and

$$AB...FG...H\overline{C}, AB...FG...H\overline{D}, ..., AB...FG...H\overline{E}$$

All axioms of the first group have the same length and differ only by one literal. The set of differing literals forms the right part of equivalence if we replace signs of these literals by opposite.

The second group has the same properties , but in this group the set of differing literals form the left part of the equivalence (to within signs of literals).

This property is used in the next rule.

Rule 4.17 of detection of equivalencies in a closed theory:

- we find axioms differing in one literal and construct the new (temporary) axiom containing common part of found axioms and all differing literals, differing literals are marked in a new axiom;

- we compare new axioms to detect pairs such that every their literal with the same atom satisfies one of the next conditions:

1) both are equal,
2) both are marked,
3) one literal is marked, the other is absent;
4) both differ by sign, but one of them is marked;

- we select from these pairs such that they have two or more literals with property 4, but one of the marked literals must be in the first axiom of a pair and the other marked literal must be in the second axiom.

If such pairs do exist then the theory contains equivalencies.

This rule has two exceptions.

If the left and right part of equivalence have one literal, then among initial axioms there must be two axioms with the same length and differing only in the sign of one of the literals (i.e. containing one contrary pairs of literals and with the other literals to be equal). Therefore if such pair of axioms exists, then theory has equivalence.

The next exception is: if one (and only one) part of equivalence contains one literal, then there must be a pair of axioms such that one of these axioms is initial, the other is new. All literals of the axioms must have property 1 or 3, except two literals differing by the sign. If such the pair exists then the theory has equivalence.

Theories with equivalence must be removed.

Rule 4.18 of detection of subsumption in a closed theory:

- we construct the set of all resolutions;

- if a resolvent of any resolution is absent among axioms of the theory, then the theory has subsumption of this resolvent by one of the axioms.

Theories with subsumptions should be removed.

Rule 4.19 of detection of superfluous axioms in closed theories:

- we construct the set of all resolutions;

- if an axiom is not present in resolutions (as an initial axiom or as a resolvent) then this axiom is superfluous in the theory.

Rule 4.20 of detection of passive literals in a closed theory:

- for every axioms we construct the list of possible passive literals without signs (i.e. the list of atoms), before iteration all lists contain all the atoms of the theory;

- we construct the set of all resolutions;

- simultaneously we correct the lists of atoms, the list of atoms for a resolvent becomes empty, in the list of atoms for every resolution arguments we remove all atoms not to meet in both resolution arguments with the same sign;

- after constructing the set of all resolutions we delete two atoms in the list of superfluous atoms if both atoms and only they form contrary pairs with any axiom of the theory (resolution of axioms with two contrary pairs is impossible but after deleting a superfluous literal two contrary pairs can transform into one and resolution becomes possible).

If after executing this rule we have an axiom with a non-empty list of atoms then the theory has superfluous literals (in this axiom).

Rule 4.21 of detection of passive axioms in closed theory:

- we construct the set of all resolutions;

- simultaneously we construct the list of resolutions for every axiom if this axiom is an argument or resolvent of the resolutions;

- we seek an axiom with the list of resolutions such that this list not to subsume the other lists (i.e. any list has a resolution not to meet in the list of this axiom);

- if such axiom exists then this axiom is passive (if we delete it then the theory remains closed).

Appendix. Theories with Passive Axioms (n=5).

Below we give only non-automorphic independent axiom systems.

1. $ABCD, A\overline{BCD}, AC\overline{D}E, A\overline{C}D\overline{E} \vdash ABCE, ABD\overline{E}, A\overline{BCE}, A\overline{BD}E.$
   $ABCD, A\overline{BCE}, A\overline{BD}E, AC\overline{D}E, A\overline{C}D\overline{E} \vdash ABCE, ABD\overline{E}, A\overline{BCD}.$
2. $ABCD, A\overline{BCD}, AC\overline{D}E, A\overline{C}D\overline{E}, BCDE \vdash ABCE, ABD\overline{E}, A\overline{BCE}, A\overline{BD}E.$
   $ABCD, A\overline{BCE}, A\overline{BD}E, AC\overline{D}E, A\overline{C}D\overline{E}, BCDE \vdash ABCE, ABD\overline{E}, A\overline{BCD}.$
   $ABD\overline{E}, A\overline{BCD}, A\overline{BCE}, AC\overline{D}E, BCDE \vdash ABCD, ABCE, A\overline{BD}E, A\overline{C}D\overline{E}.$
   $ABD\overline{E}, A\overline{BCE}, A\overline{BD}E, AC\overline{D}E, BCDE \vdash ABCD, ABCE, A\overline{BCD}, A\overline{C}D\overline{E}.$
   $ABCD, ABCE, A\overline{BCD}, A\overline{BD}E, A\overline{C}D\overline{E}, BCDE \vdash ABD\overline{E}, A\overline{BCE}, AC\overline{D}E.$
   $ABCE, ABD\overline{E}, A\overline{BCD}, A\overline{BCE}, AC\overline{D}E, BCDE \vdash ABCD, A\overline{BD}E, A\overline{C}D\overline{E}.$
3. $ABC, A\overline{BC}D, A\overline{BDE}, B\overline{C}DE \vdash ABDE, A\overline{BCE}, AC\overline{DE}, A\overline{C}DE.$
   $ABC, A\overline{BCE}, A\overline{BDE}, A\overline{C}DE, B\overline{C}DE \vdash ABDE, A\overline{BC}D, AC\overline{DE}.$
   $ABC, A\overline{BC}D, A\overline{BCE}, AC\overline{DE}, B\overline{C}DE \vdash ABDE, A\overline{BDE}, A\overline{C}DE.$
   $ABC, A\overline{BCE}, AC\overline{DE}, A\overline{C}DE, B\overline{C}DE \vdash ABDE, A\overline{BC}D, A\overline{BDE}.$
4. $ABCD, A\overline{BCD}, AC\overline{D}E, A\overline{C}D\overline{E}, BC\overline{D}E, B\overline{C}D\overline{E} \vdash ABCE, ABD\overline{E}, A\overline{BCE}, A\overline{BD}E.$
   $ABCD, A\overline{BCE}, A\overline{BD}E, BC\overline{D}E, B\overline{C}D\overline{E} \vdash ABCE, ABD\overline{E}, A\overline{BCD}, AC\overline{D}E, A\overline{C}D\overline{E}.$
   $ABCD, A\overline{BCD}, A\overline{BCE}, AC\overline{D}E, BC\overline{D}E, B\overline{C}D\overline{E} \vdash ABCE, ABD\overline{E}, A\overline{BD}E, A\overline{C}D\overline{E}.$
   $ABCD, A\overline{BCE}, A\overline{BD}E, A\overline{C}D\overline{E}, BC\overline{D}E, B\overline{C}D\overline{E} \vdash ABCE, ABD\overline{E}, A\overline{BCD}, AC\overline{D}E.$
   $ABCE, ABD\overline{E}, A\overline{BCE}, A\overline{BD}E, BC\overline{D}E, B\overline{C}D\overline{E} \vdash ABCD, A\overline{BCD}, AC\overline{D}E, A\overline{C}D\overline{E}.$
   $ABCE, ABD\overline{E}, A\overline{BCD}, A\overline{BD}E, A\overline{C}D\overline{E}, BC\overline{D}E, B\overline{C}D\overline{E} \vdash ABCD, A\overline{BCE}, AC\overline{D}E.$
   $ABCE, ABD\overline{E}, A\overline{BCD}, AC\overline{D}E, A\overline{C}D\overline{E}, BC\overline{D}E, B\overline{C}D\overline{E} \vdash ABCD, A\overline{BCE}, A\overline{BD}E.$
5. $ABCD, A\overline{BCD}, AC\overline{D}E, A\overline{C}D\overline{E}, B\overline{C}D\overline{E}, \overline{B}C\overline{D}E \vdash ABCE, ABD\overline{E}, A\overline{BCE}, A\overline{BD}E.$
   $ABCD, A\overline{BCE}, A\overline{BD}E, AC\overline{D}E, B\overline{C}D\overline{E}, \overline{B}C\overline{D}E \vdash ABCE, ABD\overline{E}, A\overline{BCD}, A\overline{C}D\overline{E}.$
   $ABCD, A\overline{BCD}, A\overline{BCE}, AC\overline{D}E, B\overline{C}D\overline{E}, \overline{B}C\overline{D}E \vdash ABCE, ABD\overline{E}, A\overline{BD}E, A\overline{C}D\overline{E}.$
   $ABCD, ABCE, A\overline{BCD}, A\overline{BCE}, B\overline{C}D\overline{E}, \overline{B}C\overline{D}E \vdash ABD\overline{E}, A\overline{BD}E, AC\overline{D}E, A\overline{C}D\overline{E}.$
6. $AB\overline{CD}, A\overline{B}C\overline{E}, AC\overline{DE}, \overline{A}BDE, \overline{AB}CD, B\overline{C}DE \vdash AB\overline{C}E, AB\overline{DE}, \overline{A}CDE, \overline{B}CD\overline{E}.$
   $AB\overline{CD}, A\overline{B}C\overline{E}, AC\overline{DE}, \overline{AB}CD, \overline{A}CDE, B\overline{C}DE \vdash AB\overline{C}E, AB\overline{DE}, \overline{A}BDE, \overline{B}CD\overline{E}.$
   $AB\overline{CD}, AC\overline{DE}, \overline{A}CDE, B\overline{C}DE, \overline{B}CD\overline{E} \vdash AB\overline{C}E, AB\overline{DE}, A\overline{B}C\overline{E}, \overline{A}BDE, \overline{AB}CD.$
   $AB\overline{CD}, AB\overline{C}E, AC\overline{DE}, \overline{A}BDE, \overline{AB}CD, \overline{B}CD\overline{E} \vdash AB\overline{DE}, A\overline{B}C\overline{E}, \overline{A}CDE, B\overline{C}DE.$
   $AB\overline{CD}, AB\overline{C}E, AC\overline{DE}, \overline{A}BDE, \overline{A}CDE, \overline{B}CD\overline{E} \vdash AB\overline{DE}, A\overline{B}C\overline{E}, \overline{AB}CD, B\overline{C}DE.$
   $AB\overline{C}E, AB\overline{DE}, A\overline{B}C\overline{E}, \overline{A}BDE, \overline{A}CDE, \overline{B}CD\overline{E} \vdash AB\overline{CD}, AC\overline{DE}, \overline{AB}CD, B\overline{C}DE.$
   $AB\overline{C}E, AB\overline{DE}, AC\overline{DE}, \overline{A}BDE, \overline{A}CDE, \overline{B}CD\overline{E} \vdash AB\overline{CD}, A\overline{B}C\overline{E}, \overline{AB}CD, B\overline{C}DE.$
7. $A\overline{B}CD, AC\overline{D}E, A\overline{C}D\overline{E}, \overline{A}B\overline{CE}, \overline{A}B\overline{D}E \vdash A\overline{B}CE, A\overline{B}D\overline{E}, \overline{A}B\overline{CD}, BC\overline{D}E, B\overline{C}D\overline{E}.$
   $A\overline{B}CD, AC\overline{D}E, A\overline{C}D\overline{E}, \overline{A}B\overline{CD}, \overline{A}B\overline{D}E, B\overline{C}D\overline{E} \vdash A\overline{B}CE, A\overline{B}D\overline{E}, \overline{A}B\overline{CE}, BC\overline{D}E.$
   $A\overline{B}CD, AC\overline{D}E, A\overline{C}D\overline{E}, \overline{A}B\overline{CD}, BC\overline{D}E, B\overline{C}D\overline{E} \vdash A\overline{B}CE, A\overline{B}D\overline{E}, \overline{A}B\overline{CE}, \overline{A}B\overline{D}E.$
   $A\overline{B}CD, A\overline{B}D\overline{E}, AC\overline{D}E, \overline{A}B\overline{CD}, \overline{A}B\overline{D}E, B\overline{C}D\overline{E} \vdash A\overline{B}CE, A\overline{C}D\overline{E}, \overline{A}B\overline{CE}, BC\overline{D}E.$
   $A\overline{B}CD, A\overline{B}D\overline{E}, AC\overline{D}E, \overline{A}B\overline{CE}, \overline{A}B\overline{D}E, B\overline{C}D\overline{E} \vdash A\overline{B}CE, A\overline{C}D\overline{E}, \overline{A}B\overline{CD}, BC\overline{D}E.$
   $A\overline{B}CE, A\overline{B}D\overline{E}, \overline{A}B\overline{CE}, \overline{A}B\overline{D}E, BC\overline{D}E, B\overline{C}D\overline{E} \vdash A\overline{B}CD, AC\overline{D}E, A\overline{C}D\overline{E}, \overline{A}B\overline{CD}.$
   $A\overline{B}CE, A\overline{B}D\overline{E}, A\overline{C}D\overline{E}, \overline{A}B\overline{CE}, \overline{A}B\overline{D}E, BC\overline{D}E \vdash A\overline{B}CD, AC\overline{D}E, \overline{A}B\overline{CD}, B\overline{C}D\overline{E}.$
   $A\overline{B}CE, A\overline{B}D\overline{E}, AC\overline{D}E, \overline{A}B\overline{CE}, \overline{A}B\overline{D}E, B\overline{C}D\overline{E} \vdash A\overline{B}CD, A\overline{C}D\overline{E}, \overline{A}B\overline{CD}, BC\overline{D}E.$
8. $ABCD, AC\overline{D}E, \overline{ABCD}, \overline{ABD}E, \overline{AC}D\overline{E}, B\overline{C}D\overline{E} \vdash ABCE, ABD\overline{E}, \overline{ABCE}, \overline{B}C\overline{D}E.$
   $ABCD, AC\overline{D}E, \overline{ABCE}, \overline{ABD}E, B\overline{C}D\overline{E} \vdash ABCE, ABD\overline{E}, \overline{ABCD}, \overline{AC}D\overline{E}, \overline{B}C\overline{D}E.$
   $ABCD, AC\overline{D}E, \overline{ABCD}, \overline{AC}D\overline{E}, B\overline{C}D\overline{E}, \overline{B}C\overline{D}E \vdash ABCE, ABD\overline{E}, \overline{ABCE}, \overline{ABD}E.$
   $ABCD, ABD\overline{E}, AC\overline{D}E, \overline{ABCD}, \overline{ABD}E, \overline{AC}D\overline{E} \vdash ABCE, \overline{ABCE}, B\overline{C}D\overline{E}, \overline{B}C\overline{D}E.$
9. $ABC, A\overline{BC}D, A\overline{BD}E, B\overline{C}D\overline{E}, \overline{B}C\overline{D}E \vdash ABD\overline{E}, A\overline{BC}E, AC\overline{D}E, A\overline{C}D\overline{E}.$
   $ABC, A\overline{BC}D, A\overline{BC}E, B\overline{C}D\overline{E}, \overline{B}C\overline{D}E \vdash ABD\overline{E}, A\overline{BD}E, AC\overline{D}E, A\overline{C}D\overline{E}.$
10. $ABC, A\overline{BC}D, A\overline{BD}E, B\overline{C}D\overline{E}, \overline{BCD}E \vdash ABD\overline{E}, A\overline{BC}E, AC\overline{D}E, A\overline{C}D\overline{E}.$
    $ABC, A\overline{BC}E, A\overline{BD}E, A\overline{C}D\overline{E}, B\overline{C}D\overline{E}, \overline{BCD}E \vdash ABD\overline{E}, A\overline{BC}D, AC\overline{D}E.$
    $ABC, A\overline{BC}D, AC\overline{D}E, B\overline{C}D\overline{E}, \overline{BCD}E \vdash ABD\overline{E}, A\overline{BC}E, A\overline{BD}E, A\overline{C}D\overline{E}.$
    $ABC, A\overline{BC}E, AC\overline{D}E, A\overline{C}D\overline{E}, B\overline{C}D\overline{E}, \overline{BCD}E \vdash ABD\overline{E}, A\overline{BC}D, A\overline{BD}E.$
11. $ABC, A\overline{BC}E, A\overline{CD}E, \overline{A}BD\overline{E}, \overline{ABC}D \vdash AB\overline{D}E, \overline{AC}D\overline{E}, BCD\overline{E}, \overline{BC}DE.$
    $ABC, A\overline{CD}E, \overline{A}BD\overline{E}, \overline{ABC}D, \overline{BC}DE \vdash AB\overline{D}E, A\overline{BC}E, \overline{AC}D\overline{E}, BCD\overline{E}.$
    $ABC, A\overline{CD}E, \overline{A}BD\overline{E}, \overline{AC}D\overline{E}, \overline{BC}DE \vdash AB\overline{D}E, A\overline{BC}E, \overline{ABC}D, BCD\overline{E}.$
    $ABC, A\overline{BC}E, A\overline{CD}E, \overline{ABC}D, \overline{AC}D\overline{E}, BCD\overline{E} \vdash AB\overline{D}E, \overline{A}BD\overline{E}, \overline{BC}DE.$
    $ABC, A\overline{CD}E, \overline{AC}D\overline{E}, BCD\overline{E}, \overline{BC}DE \vdash AB\overline{D}E, A\overline{BC}E, \overline{A}BD\overline{E}, \overline{ABC}D.$
    $ABC, AB\overline{D}E, A\overline{BC}E, \overline{A}BD\overline{E}, \overline{ABC}D \vdash A\overline{CD}E, \overline{AC}D\overline{E}, BCD\overline{E}, \overline{BC}DE.$
    $ABC, AB\overline{D}E, A\overline{BC}E, \overline{A}BD\overline{E}, \overline{AC}D\overline{E}, \overline{BC}DE \vdash A\overline{CD}E, \overline{ABC}D, BCD\overline{E}.$
    $ABC, AB\overline{D}E, A\overline{BC}E, \overline{ABC}D, \overline{AC}D\overline{E}, BCD\overline{E} \vdash A\overline{CD}E, \overline{A}BD\overline{E}, \overline{BC}DE.$
    $ABC, AB\overline{D}E, A\overline{BC}E, \overline{AC}D\overline{E}, BCD\overline{E}, \overline{BC}DE \vdash A\overline{CD}E, \overline{A}BD\overline{E}, \overline{ABC}D.$
12. $ABC, A\overline{BD}E, \overline{ABC}D, \overline{ABC}E, B\overline{C}D\overline{E} \vdash ABD\overline{E}, AC\overline{D}E, \overline{AC}D\overline{E}, \overline{BCD}E.$
    $ABC, A\overline{BD}E, \overline{ABC}E, \overline{AC}D\overline{E}, B\overline{C}D\overline{E} \vdash ABD\overline{E}, AC\overline{D}E, \overline{ABC}D, \overline{BCD}E.$
    $ABC, A\overline{BD}E, \overline{ABC}D, B\overline{C}D\overline{E}, \overline{BCD}E \vdash ABD\overline{E}, AC\overline{D}E, \overline{ABC}E, \overline{AC}D\overline{E}.$
    $ABC, ABD\overline{E}, A\overline{BD}E, \overline{ABC}E, \overline{AC}D\overline{E} \vdash AC\overline{D}E, \overline{ABC}D, B\overline{C}D\overline{E}, \overline{BCD}E.$
    $ABC, ABD\overline{E}, A\overline{BD}E, \overline{ABC}D, \overline{AC}D\overline{E}, \overline{BCD}E \vdash AC\overline{D}E, \overline{ABC}E, B\overline{C}D\overline{E}.$
    $ABC, AC\overline{D}E, \overline{ABC}D, B\overline{C}D\overline{E}, \overline{BCD}E \vdash ABD\overline{E}, A\overline{BD}E, \overline{ABC}E, \overline{AC}D\overline{E}.$
    $ABC, AC\overline{D}E, \overline{ABC}E, \overline{AC}D\overline{E}, B\overline{C}D\overline{E}, \overline{BCD}E \vdash ABD\overline{E}, A\overline{BD}E, \overline{ABC}D.$

$ABC, ABD\overline{E}, AC\overline{D}E, \overline{ABC}D, \overline{AC}D\overline{E}, \overline{BCD}E \vdash A\overline{BD}E, \overline{ABC}E, B\overline{C}D\overline{E}.$
$ABC, ABD\overline{E}, AC\overline{D}E, \overline{ABC}E, \overline{AC}D\overline{E}, \overline{BCD}E \vdash A\overline{BD}E, \overline{ABC}D, B\overline{C}D\overline{E}.$
13. $ABC, A\overline{BC}D, A\overline{B}D\overline{E}, \overline{A}B\overline{D}E, \overline{AC}DE \vdash ACD\overline{E}, \overline{A}B\overline{C}E, BC\overline{D}E, \overline{BC}DE.$
$ABC, A\overline{B}D\overline{E}, \overline{A}B\overline{D}E, \overline{AC}DE, \overline{BC}DE \vdash A\overline{BC}D, ACD\overline{E}, \overline{A}B\overline{C}E, BC\overline{D}E.$
$ABC, A\overline{BC}D, A\overline{B}D\overline{E}, \overline{A}B\overline{C}E, \overline{AC}DE, BC\overline{D}E \vdash ACD\overline{E}, \overline{A}B\overline{D}E, \overline{BC}DE.$
$ABC, A\overline{B}D\overline{E}, \overline{A}B\overline{C}E, BC\overline{D}E, \overline{BC}DE \vdash A\overline{BC}D, ACD\overline{E}, \overline{A}B\overline{D}E, \overline{AC}DE.$
$ABC, A\overline{BC}D, ACD\overline{E}, \overline{A}B\overline{C}E, \overline{A}B\overline{D}E, \overline{BC}DE \vdash A\overline{B}D\overline{E}, \overline{AC}DE, BC\overline{D}E.$
$ABC, A\overline{BC}D, ACD\overline{E}, \overline{A}B\overline{C}E, \overline{AC}DE, BC\overline{D}E \vdash A\overline{B}D\overline{E}, \overline{A}B\overline{D}E, \overline{BC}DE.$
14. $ABC, A\overline{BC}E, A\overline{CD}E, \overline{ABC}D, \overline{A}CD\overline{E} \vdash AB\overline{D}E, \overline{AB}D\overline{E}, BCD\overline{E}, \overline{BC}DE.$
$ABC, A\overline{CD}E, \overline{ABC}D, \overline{A}CD\overline{E}, \overline{BC}DE \vdash AB\overline{D}E, A\overline{BC}E, \overline{AB}D\overline{E}, BCD\overline{E}.$
$ABC, A\overline{CD}E, \overline{AB}D\overline{E}, \overline{A}CD\overline{E}, \overline{BC}DE \vdash AB\overline{D}E, A\overline{BC}E, \overline{ABC}D, BCD\overline{E}.$
$ABC, A\overline{BC}E, A\overline{CD}E, \overline{ABC}D, \overline{AB}D\overline{E}, BCD\overline{E} \vdash AB\overline{D}E, \overline{A}CD\overline{E}, \overline{BC}DE.$
$ABC, A\overline{CD}E, \overline{AB}D\overline{E}, BCD\overline{E}, \overline{BC}DE \vdash AB\overline{D}E, A\overline{BC}E, \overline{ABC}D, \overline{A}CD\overline{E}.$
$ABC, AB\overline{D}E, A\overline{BC}E, \overline{ABC}D, \overline{A}CD\overline{E} \vdash A\overline{CD}E, \overline{AB}D\overline{E}, BCD\overline{E}, \overline{BC}DE.$
$ABC, AB\overline{D}E, A\overline{BC}E, \overline{AB}D\overline{E}, \overline{A}CD\overline{E}, \overline{BC}DE \vdash A\overline{CD}E, \overline{ABC}D, BCD\overline{E}.$
$ABC, AB\overline{D}E, A\overline{BC}E, \overline{ABC}D, \overline{AB}D\overline{E}, BCD\overline{E} \vdash A\overline{CD}E, \overline{A}CD\overline{E}, \overline{BC}DE.$
$ABC, AB\overline{D}E, A\overline{BC}E, \overline{AB}D\overline{E}, BCD\overline{E}, \overline{BC}DE \vdash A\overline{CD}E, \overline{ABC}D, \overline{A}CD\overline{E}.$
15. $ABC, \overline{ABC}D, \overline{ABD}E, B\overline{C}D\overline{E}, \overline{B}C\overline{D}E \vdash ABD\overline{E}, AC\overline{D}E, \overline{ABC}E, \overline{A}CD\overline{E}.$
$ABC, \overline{ABC}D, \overline{ABC}E, B\overline{C}D\overline{E}, \overline{B}C\overline{D}E \vdash ABD\overline{E}, AC\overline{D}E, \overline{ABD}E, \overline{A}CD\overline{E}.$
$ABC, AC\overline{D}E, \overline{ABC}D, \overline{ABD}E, B\overline{C}D\overline{E} \vdash ABD\overline{E}, \overline{ABC}E, \overline{A}CD\overline{E}, \overline{B}C\overline{D}E.$
$ABC, AC\overline{D}E, \overline{ABC}E, \overline{ABD}E, \overline{A}CD\overline{E}, B\overline{C}D\overline{E} \vdash ABD\overline{E}, \overline{ABC}D, \overline{B}C\overline{D}E.$
$ABC, ABD\overline{E}, AC\overline{D}E, \overline{ABC}D, \overline{ABD}E, \overline{AC}D\overline{E} \vdash \overline{ABC}E, B\overline{C}D\overline{E}, \overline{B}C\overline{D}E.$
16. $ABC, \overline{AB}D, A\overline{BC}E, A\overline{BD}E, \overline{A}CD\overline{E} \vdash AC\overline{D}E, BCD\overline{E}, \overline{BC}DE.$
$ABC, \overline{AB}D, A\overline{BD}E, \overline{A}CD\overline{E}, \overline{BC}DE \vdash A\overline{BC}E, AC\overline{D}E, BCD\overline{E}.$
$ABC, \overline{AB}D, A\overline{BD}E, BCD\overline{E}, \overline{BC}DE \vdash A\overline{BC}E, AC\overline{D}E, \overline{A}CD\overline{E}.$
$ABC, \overline{AB}D, A\overline{BC}E, AC\overline{D}E, BCD\overline{E} \vdash A\overline{BD}E, \overline{A}CD\overline{E}, \overline{BC}DE.$
17. $ABC, A\overline{BCE}, A\overline{B}D\overline{E}, \overline{A}B\overline{C}E, \overline{ABC}D, B\overline{CD}E \vdash AB\overline{D}E, ACD\overline{E}, \overline{AC}DE, \overline{BC}D\overline{E}.$
$ABC, A\overline{BCE}, A\overline{B}D\overline{E}, \overline{ABC}D, \overline{AC}DE, B\overline{CD}E \vdash AB\overline{D}E, ACD\overline{E}, \overline{A}B\overline{C}E, \overline{BC}D\overline{E}.$
$ABC, A\overline{BCE}, A\overline{B}D\overline{E}, \overline{AC}DE, B\overline{CD}E, \overline{BC}D\overline{E} \vdash AB\overline{D}E, ACD\overline{E}, \overline{A}B\overline{C}E, \overline{ABC}D.$
$ABC, AB\overline{D}E, A\overline{BCE}, A\overline{BDE}, \overline{A}B\overline{CE}, \overline{ABCD} \vdash ACD\overline{E}, \overline{AC}DE, B\overline{CD}E, \overline{BC}D\overline{E}.$
$ABC, AB\overline{D}E, A\overline{BCE}, A\overline{BDE}, \overline{A}B\overline{CE}, \overline{AC}DE, \overline{BC}D\overline{E} \vdash ACD\overline{E}, \overline{ABCD}, B\overline{CD}E.$
$ABC, A\overline{BCE}, ACD\overline{E}, \overline{A}B\overline{C}E, \overline{ABC}D, B\overline{CD}E \vdash AB\overline{D}E, A\overline{B}D\overline{E}, \overline{AC}DE, \overline{BC}D\overline{E}.$
$ABC, A\overline{BCE}, ACD\overline{E}, \overline{ABC}D, \overline{AC}DE, B\overline{CD}E \vdash AB\overline{D}E, A\overline{B}D\overline{E}, \overline{A}B\overline{C}E, \overline{BC}D\overline{E}.$
$ABC, A\overline{BCE}, ACD\overline{E}, \overline{AC}DE, B\overline{CD}E, \overline{BC}D\overline{E} \vdash AB\overline{D}E, A\overline{B}D\overline{E}, \overline{A}B\overline{C}E, \overline{ABC}D.$
$ABC, AB\overline{D}E, A\overline{BCE}, ACD\overline{E}, \overline{A}B\overline{CE}, \overline{ABCD} \vdash A\overline{BDE}, \overline{AC}DE, B\overline{CD}E, \overline{BC}D\overline{E}.$
$ABC, AB\overline{D}E, A\overline{BCE}, ACD\overline{E}, \overline{A}B\overline{CE}, \overline{AC}DE, \overline{BC}D\overline{E} \vdash A\overline{BDE}, \overline{ABCD}, B\overline{CD}E.$
18. $ABC, \overline{AB}D, \overline{A}B\overline{DE}, \overline{AC}DE, \overline{B}CD\overline{E} \vdash ACD\overline{E}, \overline{ABC}E, BC\overline{D}E.$
$ABC, \overline{AB}D, \overline{A}B\overline{DE}, \overline{ABC}E, \overline{B}CD\overline{E} \vdash ACD\overline{E}, \overline{AC}DE, BC\overline{D}E.$
$ABC, \overline{AB}D, ACD\overline{E}, \overline{A}B\overline{D}E, \overline{ACD}E \vdash \overline{ABC}E, BC\overline{D}E, \overline{B}CD\overline{E}.$
$ABC, \overline{AB}D, ACD\overline{E}, \overline{A}B\overline{D}E, \overline{ABC}E \vdash \overline{ACD}E, BC\overline{D}E, \overline{B}CD\overline{E}.$
$ABC, \overline{AB}D, \overline{ACD}E, BC\overline{D}E, \overline{B}CD\overline{E} \vdash ACD\overline{E}, \overline{ABD}E, \overline{ABC}E.$
$ABC, \overline{AB}D, ACD\overline{E}, \overline{ACD}E, BC\overline{D}E \vdash \overline{ABD}E, \overline{ABC}E, \overline{B}CD\overline{E}.$
19. $ABD, \overline{A}CE, \overline{ABDE}, B\overline{CDE} \vdash AB\overline{CE}, \overline{ABCD}, \overline{ACDE}, BCDE.$
$ABD, \overline{A}CE, \overline{ABCD}, \overline{ACDE}, B\overline{CDE} \vdash AB\overline{CE}, \overline{ABDE}, BCDE.$
$ABD, \overline{A}CE, AB\overline{CE}, \overline{ABDE}, \overline{ACDE} \vdash \overline{ABCD}, BCDE, B\overline{CDE}.$
$ABD, \overline{A}CE, AB\overline{CE}, \overline{ABCD}, \overline{ACDE} \vdash \overline{ABDE}, BCDE, B\overline{CDE}.$
20. $ABD, \overline{A}CE, \overline{ACDE}, \overline{ABDE} \vdash AB\overline{CE}, \overline{ABCD}, BCDE, B\overline{CDE}.$
$ABD, \overline{A}CE, \overline{ACDE}, \overline{ABCD}, B\overline{CDE} \vdash AB\overline{CE}, \overline{ABDE}, BCDE.$
$ABD, \overline{A}CE, AB\overline{CE}, \overline{ABCD}, B\overline{CDE} \vdash \overline{ACDE}, \overline{ABDE}, BCDE.$
21. $ABC, \overline{AB}D, A\overline{C}DE, \overline{B}C\overline{DE} \vdash ABDE, AC\overline{DE}, \overline{ABCE}, \overline{BC}DE.$
$ABC, \overline{AB}D, A\overline{C}DE, \overline{AC}DE, \overline{ABCE} \vdash ABDE, \overline{B}C\overline{DE}, \overline{BC}DE.$
$ABC, \overline{AB}D, ABDE, AC\overline{DE}, \overline{ABCE}, \overline{BCD}E \vdash \overline{AC}DE, \overline{BCDE}.$
22. $ABC, \overline{AB}D, A\overline{C}DE, \overline{B}CD\overline{E}, \overline{BCD}E \vdash AB\overline{D}E, ACD\overline{E}, \overline{ABC}E.$
$ABC, \overline{AB}D, A\overline{C}DE, \overline{ABC}E, \overline{B}C\overline{DE} \vdash AB\overline{D}E, ACD\overline{E}, \overline{BC}D\overline{E}.$
$ABC, \overline{AB}D, ACD\overline{E}, \overline{ACD}E, \overline{ABCE} \vdash AB\overline{D}E, \overline{B}CD\overline{E}, \overline{BC}DE.$
$ABC, \overline{AB}D, AB\overline{D}E, ACD\overline{E}, \overline{BCDE} \vdash \overline{ACD}E, \overline{ABC}E, BC\overline{D}E.$
23. $ABC, \overline{ABC}, A\overline{C}DE, \overline{B}C\overline{DE} \vdash ABDE, AC\overline{DE}, \overline{ABDE}, \overline{BC}DE.$
$ABC, \overline{ABC}, A\overline{C}DE, \overline{AC}DE, \overline{ABDE} \vdash ABDE, \overline{B}C\overline{DE}, \overline{BC}DE.$
$ABC, \overline{ABC}, ABDE, \overline{B}C\overline{DE}, \overline{BC}DE \vdash AC\overline{DE}, \overline{AC}DE, \overline{ABDE}.$
$ABC, \overline{ABC}, ABDE, AC\overline{DE}, \overline{ABDE}, \overline{BC}DE \vdash \overline{AC}DE, \overline{BCDE}.$
24. $ABC, A\overline{BCD}, A\overline{BDE}, \overline{ABC}D, \overline{A}BD\overline{E}, \overline{BC}DE \vdash A\overline{BC}E, AC\overline{D}E, \overline{ACD}E, BCD\overline{E}.$
$ABC, A\overline{BCD}, A\overline{BDE}, \overline{ABDE}, \overline{AC}DE, \overline{BC}DE \vdash A\overline{BC}E, AC\overline{D}E, \overline{ABC}D, BCD\overline{E}.$
$ABC, A\overline{BCD}, A\overline{BCE}, A\overline{BDE}, \overline{A}BD\overline{E}, \overline{AC}DE \vdash AC\overline{D}E, \overline{ABC}D, BCD\overline{E}, \overline{BC}DE.$
$ABC, A\overline{BCD}, A\overline{BDE}, \overline{ABC}D, BCD\overline{E}, \overline{BC}DE \vdash A\overline{BC}E, AC\overline{D}E, \overline{ABDE}, \overline{ACD}E.$
$ABC, A\overline{BCD}, A\overline{BCE}, A\overline{BDE}, \overline{ABC}D, \overline{AC}DE, BCD\overline{E} \vdash AC\overline{D}E, \overline{ABDE}, \overline{BC}DE.$
$ABC, A\overline{BCD}, ACD\overline{E}, \overline{ABC}D, \overline{A}BD\overline{E}, \overline{BC}DE \vdash A\overline{BC}E, A\overline{BDE}, \overline{ACD}E, BCD\overline{E}.$

$ABC, A\overline{BCD}, AC\overline{D}E, \overline{A}BD\overline{E}, \overline{AC}DE, \overline{BC}DE \vdash A\overline{BC}E, A\overline{BD}E, \overline{A}B\overline{C}D, BCD\overline{E}.$
$ABC, A\overline{BCD}, A\overline{BC}E, AC\overline{D}E, \overline{A}BD\overline{E}, \overline{AC}DE \vdash A\overline{BD}E, \overline{A}B\overline{C}D, BCD\overline{E}, \overline{BC}DE.$
$ABC, A\overline{BCD}, AC\overline{D}E, \overline{A}B\overline{C}D, BCD\overline{E}, \overline{BC}DE \vdash A\overline{BC}E, A\overline{BD}E, \overline{A}BD\overline{E}, \overline{AC}DE.$
$ABC, A\overline{BCD}, A\overline{BC}E, AC\overline{D}E, \overline{A}B\overline{C}D, \overline{AC}DE, BCD\overline{E} \vdash A\overline{BD}E, \overline{A}BD\overline{E}, \overline{BC}DE.$

25. $ABC, \overline{AB}D, B\overline{C}DE, \overline{B}C\overline{DE} \vdash ABDE, AC\overline{DE}, \overline{AB}C\overline{E}, \overline{AC}DE.$
$ABC, \overline{AB}D, AC\overline{DE}, \overline{AB}C\overline{E}, B\overline{C}DE \vdash ABDE, \overline{AC}DE, \overline{B}C\overline{DE}.$
$ABC, \overline{AB}D, ABDE, \overline{AC}DE, \overline{B}C\overline{DE} \vdash AC\overline{DE}, \overline{AB}C\overline{E}, B\overline{C}DE.$
$ABC, \overline{AB}D, ABDE, AC\overline{DE}, \overline{AB}C\overline{E}, \overline{AC}DE \vdash B\overline{C}DE, \overline{B}C\overline{DE}.$

26. $ABD, \overline{AB}C, A\overline{BD}E, A\overline{C}D\overline{E}, \overline{A}CD\overline{E} \vdash AB\overline{C}E, BCD\overline{E}, \overline{B}C\overline{DE}.$
$ABD, \overline{AB}C, A\overline{CD}E, \overline{A}CD\overline{E}, \overline{B}C\overline{DE} \vdash AB\overline{C}E, A\overline{BD}E, BCD\overline{E}.$
$ABD, \overline{AB}C, A\overline{BD}E, A\overline{C}D\overline{E}, BCD\overline{E} \vdash AB\overline{C}E, \overline{A}CD\overline{E}, \overline{B}C\overline{DE}.$
$ABD, \overline{AB}C, A\overline{CD}E, BCD\overline{E}, \overline{B}C\overline{DE} \vdash AB\overline{C}E, A\overline{BD}E, \overline{A}CD\overline{E}.$
$ABD, \overline{AB}C, AB\overline{C}E, A\overline{BD}E, \overline{A}CD\overline{E} \vdash A\overline{CD}E, BCD\overline{E}, \overline{B}C\overline{DE}.$
$ABD, \overline{AB}C, AB\overline{C}E, A\overline{BD}E, BCD\overline{E} \vdash A\overline{CD}E, \overline{A}CD\overline{E}, \overline{B}C\overline{DE}.$

27. $ABC, A\overline{BC}E, A\overline{B}DE, \overline{A}B\overline{CE}, \overline{A}B\overline{DE}, \overline{ABC}D \vdash ACDE, \overline{AC}D\overline{E}, BC\overline{DE}, \overline{BC}DE.$
$ABC, A\overline{BC}E, A\overline{B}DE, \overline{A}B\overline{DE}, \overline{ABC}D, \overline{AC}D\overline{E} \vdash ACDE, \overline{A}B\overline{CE}, BC\overline{DE}, \overline{BC}DE.$
$ABC, A\overline{BC}E, A\overline{B}DE, \overline{A}B\overline{DE}, \overline{AC}D\overline{E}, \overline{BC}DE \vdash ACDE, \overline{A}B\overline{CE}, \overline{ABC}D, BC\overline{DE}.$
$ABC, A\overline{BC}E, A\overline{B}DE, \overline{A}B\overline{CE}, \overline{ABC}D, BC\overline{DE} \vdash ACDE, \overline{A}B\overline{DE}, \overline{AC}D\overline{E}, \overline{BC}DE.$
$ABC, A\overline{BC}E, A\overline{B}DE, \overline{A}B\overline{CE}, \overline{AC}D\overline{E}, BC\overline{DE}, \overline{BC}DE \vdash ACDE, \overline{A}B\overline{DE}, \overline{ABC}D.$
$ABC, A\overline{BC}E, ACDE, \overline{A}B\overline{CE}, \overline{A}B\overline{DE}, \overline{ABC}D \vdash A\overline{B}DE, \overline{AC}D\overline{E}, BC\overline{DE}, \overline{BC}DE.$
$ABC, A\overline{BC}E, ACDE, \overline{A}B\overline{DE}, \overline{ABC}D, \overline{AC}D\overline{E} \vdash A\overline{B}DE, \overline{A}B\overline{CE}, BC\overline{DE}, \overline{BC}DE.$
$ABC, A\overline{BC}E, ACDE, \overline{A}B\overline{DE}, \overline{AC}D\overline{E}, \overline{BC}DE \vdash A\overline{B}DE, \overline{A}B\overline{CE}, \overline{ABC}D, BC\overline{DE}.$
$ABC, A\overline{BC}E, ACDE, \overline{A}B\overline{CE}, \overline{ABC}D, BC\overline{DE} \vdash A\overline{B}DE, \overline{A}B\overline{DE}, \overline{AC}D\overline{E}, \overline{BC}DE.$
$ABC, A\overline{BC}E, ACDE, \overline{A}B\overline{CE}, \overline{AC}D\overline{E}, BC\overline{DE}, \overline{BC}DE \vdash A\overline{B}DE, \overline{A}B\overline{DE}, \overline{ABC}D.$

28. $ABC, A\overline{BC}E, A\overline{BD}E, \overline{A}B\overline{CE}, \overline{A}BD\overline{E}, \overline{ABC}D \vdash AC\overline{D}E, \overline{AC}D\overline{E}, BCD\overline{E}, \overline{BC}DE.$
$ABC, A\overline{BD}E, \overline{A}B\overline{CE}, \overline{A}BD\overline{E}, \overline{ABC}D, \overline{BC}DE \vdash A\overline{BC}E, AC\overline{D}E, \overline{AC}D\overline{E}, BCD\overline{E}.$
$ABC, A\overline{BD}E, \overline{A}B\overline{CE}, \overline{A}BD\overline{E}, \overline{AC}D\overline{E}, \overline{BC}DE \vdash A\overline{BC}E, AC\overline{D}E, \overline{ABC}D, BCD\overline{E}.$
$ABC, A\overline{BC}E, A\overline{BD}E, \overline{A}B\overline{CE}, \overline{ABC}D, BCD\overline{E} \vdash AC\overline{D}E, \overline{A}BD\overline{E}, \overline{AC}D\overline{E}, \overline{BC}DE.$
$ABC, A\overline{BD}E, \overline{A}B\overline{CE}, \overline{ABC}D, BCD\overline{E}, \overline{BC}DE \vdash A\overline{BC}E, AC\overline{D}E, \overline{A}BD\overline{E}, \overline{AC}D\overline{E}.$
$ABC, A\overline{BD}E, \overline{A}B\overline{CE}, \overline{AC}D\overline{E}, BCD\overline{E}, \overline{BC}DE \vdash A\overline{BC}E, AC\overline{D}E, \overline{A}BD\overline{E}, \overline{ABC}D.$
$ABC, A\overline{BC}E, AC\overline{D}E, \overline{A}B\overline{CE}, \overline{A}BD\overline{E}, \overline{ABC}D \vdash A\overline{BD}E, \overline{AC}D\overline{E}, BCD\overline{E}, \overline{BC}DE.$
$ABC, A\overline{BC}E, AC\overline{D}E, \overline{A}B\overline{CE}, \overline{A}BD\overline{E}, \overline{AC}D\overline{E}, \overline{BC}DE \vdash A\overline{BD}E, \overline{ABC}D, BCD\overline{E}.$
$ABC, A\overline{BC}E, AC\overline{D}E, \overline{A}B\overline{CE}, \overline{ABC}D, BCD\overline{E} \vdash A\overline{BD}E, \overline{A}BD\overline{E}, \overline{AC}D\overline{E}, \overline{BC}DE.$
$ABC, A\overline{BC}E, AC\overline{D}E, \overline{A}B\overline{CE}, \overline{AC}D\overline{E}, BCD\overline{E}, \overline{BC}DE \vdash A\overline{BD}E, \overline{A}BD\overline{E}, \overline{ABC}D.$

29. $ABC, \overline{AB}D, \overline{ACD}E, B\overline{CD}E, \overline{B}CD\overline{E} \vdash AB\overline{D}E, ACD\overline{E}, \overline{ABC}E.$
$ABC, \overline{AB}D, \overline{ABC}E, B\overline{CD}E, \overline{B}CD\overline{E} \vdash AB\overline{D}E, ACD\overline{E}, \overline{ACD}E.$
$ABC, \overline{AB}D, ACD\overline{E}, \overline{ABC}E, B\overline{CD}E \vdash AB\overline{D}E, \overline{ACD}E, \overline{B}CD\overline{E}.$
$ABC, \overline{AB}D, AB\overline{D}E, \overline{ACD}E, \overline{B}CD\overline{E} \vdash ACD\overline{E}, \overline{ABC}E, B\overline{CD}E.$

30. $ABC, \overline{ABC}, B\overline{C}DE, \overline{B}C\overline{DE} \vdash ABDE, AC\overline{DE}, \overline{ABD}E, \overline{AC}DE.$
$ABC, \overline{ABC}, AC\overline{DE}, \overline{ABD}E, B\overline{C}DE \vdash ABDE, \overline{AC}DE, \overline{B}C\overline{DE}.$
$ABC, \overline{ABC}, ABDE, AC\overline{DE}, \overline{ABD}E, \overline{AC}DE \vdash B\overline{C}DE, \overline{B}C\overline{DE}.$

31. $ABC, A\overline{BC}D, A\overline{C}DE, \overline{A}B\overline{CD}, \overline{A}B\overline{C}E, \overline{BCDE} \vdash ABDE, A\overline{BCE}, \overline{ACDE}, B\overline{C}DE.$
$ABC, A\overline{BC}D, A\overline{C}DE, \overline{A}B\overline{C}E, \overline{ACDE}, \overline{BCDE} \vdash ABDE, A\overline{BCE}, \overline{A}B\overline{CD}, B\overline{C}DE.$
$ABC, A\overline{BCE}, A\overline{C}DE, \overline{A}B\overline{C}E, \overline{ACDE} \vdash ABDE, A\overline{BC}D, \overline{A}B\overline{CD}, B\overline{C}DE, \overline{BCDE}.$
$ABC, A\overline{BCE}, A\overline{C}DE, \overline{A}B\overline{CD}, \overline{A}B\overline{C}E, \overline{BCDE} \vdash ABDE, A\overline{BC}D, \overline{ACDE}, B\overline{C}DE.$
$ABC, A\overline{BCE}, A\overline{C}DE, \overline{A}B\overline{CD}, \overline{ACDE}, B\overline{C}DE \vdash ABDE, A\overline{BC}D, \overline{A}B\overline{C}E, \overline{BCDE}.$
$ABC, ABDE, A\overline{BC}D, \overline{A}B\overline{CD}, \overline{A}B\overline{C}E, \overline{BCDE} \vdash A\overline{BCE}, A\overline{C}DE, \overline{ACDE}, B\overline{C}DE.$
$ABC, ABDE, A\overline{BC}D, \overline{A}B\overline{C}E, \overline{ACDE}, \overline{BCDE} \vdash A\overline{BCE}, A\overline{C}DE, \overline{A}B\overline{CD}, B\overline{C}DE.$

32. $AB, \overline{AB}CD, \overline{A}C\overline{D}E, \overline{B}CD\overline{E} \vdash ACD\overline{E}, \overline{AB}CE, BC\overline{D}E.$
$AB, ACD\overline{E}, \overline{AB}CD, \overline{A}C\overline{D}E \vdash \overline{AB}CE, BC\overline{D}E, \overline{B}CD\overline{E}.$
$AB, ACD\overline{E}, \overline{AB}CD, \overline{AB}CE, BC\overline{D}E \vdash \overline{A}C\overline{D}E, \overline{B}CD\overline{E}.$

33. $ABC, \overline{AB}D, A\overline{BC}E, A\overline{C}DE, \overline{B}C\overline{DE} \vdash ABDE, AC\overline{DE}, \overline{AB}C\overline{E}, \overline{BC}DE.$
$ABC, \overline{AB}D, A\overline{BC}E, AC\overline{DE}, A\overline{C}DE, \overline{AB}C\overline{E} \vdash ABDE, \overline{B}C\overline{DE}, \overline{BC}DE.$
$ABC, \overline{AB}D, ABDE, A\overline{BC}E, \overline{B}C\overline{DE} \vdash AC\overline{DE}, A\overline{C}DE, \overline{AB}C\overline{E}, \overline{BC}DE.$
$ABC, \overline{AB}D, ABDE, A\overline{BC}E, AC\overline{DE}, \overline{AB}C\overline{E} \vdash A\overline{C}DE, \overline{B}C\overline{DE}, \overline{BC}DE.$

34. $ABC, \overline{AB}D, A\overline{BCD}, A\overline{BC}E, A\overline{BD}E, \overline{A}CD\overline{E} \vdash AC\overline{D}E, BCD\overline{E}, \overline{BC}DE.$
$ABC, \overline{AB}D, A\overline{BCD}, A\overline{BD}E, \overline{A}CD\overline{E}, \overline{BC}DE \vdash A\overline{BC}E, AC\overline{D}E, BCD\overline{E}.$
$ABC, \overline{AB}D, A\overline{BCD}, A\overline{BD}E, BCD\overline{E}, \overline{BC}DE \vdash A\overline{BC}E, AC\overline{D}E, \overline{A}CD\overline{E}.$
$ABC, \overline{AB}D, A\overline{BCD}, AC\overline{D}E, \overline{A}CD\overline{E}, \overline{BC}DE \vdash A\overline{BC}E, A\overline{BD}E, BCD\overline{E}.$

35. $ABC, \overline{AB}D, \overline{A}B\overline{CD}, \overline{A}B\overline{D}E, \overline{ACD}E, \overline{B}CD\overline{E} \vdash ACD\overline{E}, \overline{ABC}E, BC\overline{D}E.$
$ABC, \overline{AB}D, \overline{A}B\overline{CD}, \overline{A}B\overline{D}E, \overline{ABC}E, \overline{B}CD\overline{E} \vdash ACD\overline{E}, \overline{ACD}E, BC\overline{D}E.$
$ABC, \overline{AB}D, ACD\overline{E}, \overline{A}B\overline{CD}, \overline{A}B\overline{D}E, \overline{ACD}E \vdash \overline{ABC}E, BC\overline{D}E, \overline{B}CD\overline{E}.$
$ABC, \overline{AB}D, ACD\overline{E}, \overline{A}B\overline{CD}, \overline{A}B\overline{D}E, \overline{ABC}E \vdash \overline{ACD}E, BC\overline{D}E, \overline{B}CD\overline{E}.$
$ABC, \overline{AB}D, \overline{A}B\overline{CD}, \overline{ACD}E, BC\overline{D}E, \overline{B}CD\overline{E} \vdash ACD\overline{E}, \overline{A}B\overline{D}E, \overline{ABC}E.$
$ABC, \overline{AB}D, \overline{A}B\overline{CD}, \overline{ABC}E, BC\overline{D}E, \overline{B}CD\overline{E} \vdash ACD\overline{E}, \overline{A}B\overline{D}E, \overline{ACD}E.$
$ABC, \overline{AB}D, ACD\overline{E}, \overline{A}B\overline{CD}, \overline{ACD}E, BC\overline{D}E \vdash \overline{A}B\overline{D}E, \overline{ABC}E, \overline{B}CD\overline{E}.$

$ABC, \overline{AB}D, ACD\overline{E}, \overline{A}B\overline{CD}, \overline{ABC}E, BC\overline{D}E \vdash \overline{A}B\overline{D}E, \overline{ACD}E, \overline{B}CD\overline{E}.$

36. $ABC, \overline{AB}D, A\overline{BCD}, A\overline{C}\overline{DE}, \overline{B}CD\overline{E}, \overline{BCDE} \vdash AB\overline{DE}, ACD\overline{E}, \overline{ABC}E.$
$ABC, \overline{AB}D, A\overline{BCD}, A\overline{C}\overline{DE}, \overline{ABC}E, \overline{B}CD\overline{E} \vdash AB\overline{DE}, ACD\overline{E}, \overline{BCDE}.$
$ABC, \overline{AB}D, A\overline{BCD}, ACD\overline{E}, A\overline{C}\overline{DE}, \overline{ABC}E \vdash AB\overline{DE}, \overline{B}CD\overline{E}, \overline{BCDE}.$
$ABC, \overline{AB}D, AB\overline{DE}, A\overline{BCD}, \overline{ABC}E, \overline{B}CD\overline{E} \vdash ACD\overline{E}, A\overline{C}\overline{DE}, \overline{BCDE}.$
37. $ABC, \overline{AB}D, \overline{CD}E, \overline{B}CD\overline{E} \vdash AB\overline{DE}, ACD\overline{E}, \overline{ABC}E.$
$ABC, \overline{AB}D, \overline{CD}E, ACD\overline{E} \vdash AB\overline{DE}, \overline{ABC}E, \overline{B}CD\overline{E}.$
38. $ABC, \overline{AB}D, A\overline{BC}E, A\overline{C}DE, \overline{A}C\overline{DE} \vdash ABDE, \overline{ABC}\overline{E}, BC\overline{DE}, \overline{BC}DE.$
$ABC, \overline{AB}D, A\overline{BC}E, A\overline{C}DE, \overline{ABC}\overline{E}, BC\overline{DE} \vdash ABDE, \overline{A}C\overline{DE}, \overline{BC}DE.$
$ABC, \overline{AB}D, ABDE, A\overline{BC}E, \overline{A}C\overline{DE} \vdash A\overline{C}DE, \overline{ABC}\overline{E}, BC\overline{DE}, \overline{BC}DE.$
$ABC, \overline{AB}D, ABDE, A\overline{BC}E, \overline{ABC}\overline{E}, BC\overline{DE} \vdash A\overline{C}DE, \overline{A}C\overline{DE}, \overline{BC}DE.$
39. $ABC, A\overline{BC}D, A\overline{BCE}, A\overline{B}D\overline{E}, \overline{A}B\overline{CD}, \overline{A}B\overline{DE}, \overline{AC}DE \vdash ACD\overline{E}, \overline{A}B\overline{C}E, BC\overline{DE}, \overline{BC}DE.$
$ABC, A\overline{BCE}, A\overline{B}D\overline{E}, \overline{A}B\overline{CD}, \overline{A}B\overline{DE}, \overline{AC}DE, \overline{BC}DE \vdash A\overline{BC}D, ACD\overline{E}, \overline{A}B\overline{C}E, BC\overline{DE}.$
$ABC, A\overline{BC}D, A\overline{BCE}, A\overline{B}D\overline{E}, \overline{A}B\overline{CD}, \overline{AC}DE, BC\overline{DE} \vdash ACD\overline{E}, \overline{A}B\overline{C}E, \overline{A}B\overline{DE}, \overline{BC}DE.$
$ABC, A\overline{BCE}, A\overline{B}D\overline{E}, \overline{A}B\overline{CD}, \overline{AC}DE, BC\overline{DE}, \overline{BC}DE \vdash A\overline{BC}D, ACD\overline{E}, \overline{A}B\overline{C}E, \overline{A}B\overline{DE}.$
$ABC, A\overline{BCE}, A\overline{B}D\overline{E}, \overline{A}B\overline{CD}, \overline{A}B\overline{C}E, BC\overline{DE}, \overline{BC}DE \vdash A\overline{BC}D, ACD\overline{E}, \overline{A}B\overline{DE}, \overline{AC}DE.$
$ABC, A\overline{BC}D, A\overline{BCE}, ACD\overline{E}, \overline{A}B\overline{CD}, \overline{AC}DE, BC\overline{DE} \vdash A\overline{B}D\overline{E}, \overline{A}B\overline{C}E, \overline{A}B\overline{DE}, \overline{BC}DE.$
$ABC, A\overline{BCE}, ACD\overline{E}, \overline{A}B\overline{CD}, \overline{AC}DE, BC\overline{DE}, \overline{BC}DE \vdash A\overline{BC}D, A\overline{B}D\overline{E}, \overline{A}B\overline{C}E, \overline{A}B\overline{DE}.$
40. $ABC, \overline{AB}D, A\overline{BCD}, A\overline{C}\overline{DE}, \overline{A}CD\overline{E}, \overline{BCDE} \vdash AB\overline{DE}, \overline{ABC}E, BCD\overline{E}.$
$ABC, \overline{AB}D, A\overline{BCD}, A\overline{C}\overline{DE}, \overline{ABC}E, \overline{A}CD\overline{E} \vdash AB\overline{DE}, BCD\overline{E}, \overline{BCDE}.$
$ABC, \overline{AB}D, A\overline{BCD}, A\overline{C}\overline{DE}, \overline{ABC}E, BCD\overline{E} \vdash AB\overline{DE}, \overline{A}CD\overline{E}, \overline{BCDE}.$
$ABC, \overline{AB}D, AB\overline{DE}, A\overline{BCD}, \overline{A}CD\overline{E}, \overline{BCDE} \vdash A\overline{C}\overline{DE}, \overline{ABC}E, BCD\overline{E}.$
$ABC, \overline{AB}D, AB\overline{DE}, A\overline{BCD}, \overline{ABC}E, \overline{A}CD\overline{E} \vdash A\overline{C}\overline{DE}, BCD\overline{E}, \overline{BCDE}.$
41. $ABC, A\overline{BC}D, A\overline{C}DE, \overline{A}B\overline{CD}, \overline{A}B\overline{C}E, \overline{ABDE}, \overline{BCDE} \vdash ABDE, A\overline{BCE}, \overline{ACDE}, B\overline{C}DE.$
$ABC, A\overline{BC}D, A\overline{C}DE, \overline{A}B\overline{C}E, \overline{ABDE}, \overline{ACDE}, \overline{BCDE} \vdash ABDE, A\overline{BCE}, \overline{A}B\overline{CD}, B\overline{C}DE.$
$ABC, A\overline{BCE}, A\overline{C}DE, \overline{A}B\overline{CD}, \overline{A}B\overline{C}E, \overline{ABDE} \vdash ABDE, A\overline{BC}D, \overline{ACDE}, B\overline{C}DE, \overline{BCDE}.$
$ABC, A\overline{BCE}, A\overline{C}DE, \overline{A}B\overline{C}E, \overline{ABDE}, \overline{ACDE} \vdash ABDE, A\overline{BC}D, \overline{A}B\overline{CD}, B\overline{C}DE, \overline{BCDE}.$
$ABC, A\overline{BCE}, A\overline{C}DE, \overline{A}B\overline{CD}, \overline{ABDE}, B\overline{C}DE \vdash ABDE, A\overline{BC}D, \overline{A}B\overline{C}E, \overline{ACDE}, \overline{BCDE}.$
$ABC, ABDE, A\overline{BC}D, \overline{A}B\overline{CD}, \overline{A}B\overline{C}E, \overline{ABDE}, \overline{BCDE} \vdash A\overline{BCE}, A\overline{C}DE, \overline{ACDE}, B\overline{C}DE.$
$ABC, ABDE, A\overline{BC}D, \overline{A}B\overline{C}E, \overline{ABDE}, \overline{ACDE}, \overline{BCDE} \vdash A\overline{BCE}, A\overline{C}DE, \overline{A}B\overline{CD}, B\overline{C}DE.$
$ABC, ABDE, A\overline{BC}D, A\overline{BCE}, \overline{A}B\overline{CD}, \overline{A}B\overline{C}E, \overline{ABDE} \vdash A\overline{C}DE, \overline{ACDE}, B\overline{C}DE, \overline{BCDE}.$
42. $AB, \overline{AB}CD, \overline{B}C\overline{D}E, \overline{B}CD\overline{E} \vdash AC\overline{D}E, A\overline{C}D\overline{E}, \overline{AB}CE, \overline{AB}D\overline{E}.$
$AB, \overline{AB}CE, \overline{AB}D\overline{E}, \overline{B}C\overline{D}E, \overline{B}CD\overline{E} \vdash AC\overline{D}E, A\overline{C}D\overline{E}, \overline{AB}CD.$
$AB, A\overline{C}D\overline{E}, \overline{AB}CD, \overline{AB}D\overline{E}, \overline{B}C\overline{D}E \vdash AC\overline{D}E, \overline{AB}CE, \overline{B}CD\overline{E}.$
$AB, A\overline{C}D\overline{E}, \overline{AB}CE, \overline{AB}D\overline{E}, \overline{B}C\overline{D}E \vdash AC\overline{D}E, \overline{AB}CD, \overline{B}CD\overline{E}.$
$AB, AC\overline{D}E, A\overline{C}D\overline{E}, \overline{AB}CE, \overline{AB}D\overline{E} \vdash \overline{AB}CD, \overline{B}C\overline{D}E, \overline{B}CD\overline{E}.$
43. $AB, \overline{AB}CD, \overline{AB}DE, \overline{A}C\overline{DE}, \overline{B}CDE \vdash ACDE, \overline{AB}C\overline{E}, BC\overline{DE}.$
$AB, \overline{AB}C\overline{E}, \overline{AB}DE, \overline{A}C\overline{DE}, \overline{B}CDE \vdash ACDE, \overline{AB}CD, BC\overline{DE}.$
$AB, ACDE, \overline{AB}CD, \overline{AB}DE, \overline{A}C\overline{DE} \vdash \overline{AB}C\overline{E}, BC\overline{DE}, \overline{B}CDE.$
$AB, ACDE, \overline{AB}C\overline{E}, \overline{AB}DE, \overline{A}C\overline{DE} \vdash \overline{AB}CD, BC\overline{DE}, \overline{B}CDE.$
$AB, \overline{AB}C\overline{E}, \overline{AB}DE, BC\overline{DE}, \overline{B}CDE \vdash ACDE, \overline{AB}CD, \overline{A}C\overline{DE}.$
$AB, ACDE, \overline{AB}C\overline{E}, \overline{AB}DE, BC\overline{DE} \vdash \overline{AB}CD, \overline{A}C\overline{DE}, \overline{B}CDE.$
44. $AB, \overline{AB}DE, \overline{A}C\overline{DE}, \overline{B}CD\overline{E} \vdash ACD\overline{E}, \overline{AB}CD, \overline{AB}CE, BC\overline{DE}.$
$AB, \overline{AB}CE, \overline{AB}DE, \overline{A}C\overline{DE}, \overline{B}CD\overline{E} \vdash ACD\overline{E}, \overline{AB}CD, BC\overline{DE}.$
$AB, ACD\overline{E}, \overline{AB}CD, \overline{AB}DE, \overline{A}C\overline{DE} \vdash \overline{AB}CE, BC\overline{DE}, \overline{B}CD\overline{E}.$
$AB, ACD\overline{E}, \overline{AB}CD, \overline{AB}CE, \overline{AB}DE, BC\overline{DE} \vdash \overline{A}C\overline{DE}, \overline{B}CD\overline{E}.$
45. $AB, \overline{AB}CD, \overline{A}CD\overline{E}, \overline{B}C\overline{D}E \vdash AC\overline{D}E, \overline{AB}CE, \overline{AB}D\overline{E}, B\overline{C}D\overline{E}.$
$AB, \overline{AB}CE, \overline{AB}D\overline{E}, \overline{A}CD\overline{E}, \overline{B}C\overline{D}E \vdash AC\overline{D}E, \overline{AB}CD, B\overline{C}D\overline{E}.$
$AB, AC\overline{D}E, \overline{AB}CD, \overline{AB}CE, \overline{A}CD\overline{E} \vdash \overline{AB}D\overline{E}, B\overline{C}D\overline{E}, \overline{B}C\overline{D}E.$
$AB, AC\overline{D}E, \overline{AB}CE, \overline{AB}D\overline{E}, \overline{A}CD\overline{E} \vdash \overline{AB}CD, B\overline{C}D\overline{E}, \overline{B}C\overline{D}E.$
$AB, AC\overline{D}E, \overline{AB}CE, \overline{AB}D\overline{E}, B\overline{C}D\overline{E} \vdash \overline{AB}CD, \overline{A}CD\overline{E}, \overline{B}C\overline{D}E.$
46. $ABE, AC\overline{E}, BD\overline{E}, CDE \vdash ABC, ABD, ACD, BCD.$
47. $ABC, \overline{AB}D, A\overline{BC}E, A\overline{BDE}, A\overline{C}DE, \overline{B}C\overline{DE} \vdash ABDE, A\overline{BCD}, AC\overline{DE}, \overline{ABC}\overline{E}, \overline{BC}DE.$
$ABC, \overline{AB}D, A\overline{BC}E, A\overline{BDE}, A\overline{C}DE, \overline{ABC}\overline{E} \vdash ABDE, A\overline{BCD}, AC\overline{DE}, \overline{B}C\overline{DE}, \overline{BC}DE.$
$ABC, \overline{AB}D, A\overline{BCD}, A\overline{C}DE, \overline{B}C\overline{DE} \vdash ABDE, A\overline{BC}E, A\overline{BDE}, AC\overline{DE}, \overline{ABC}\overline{E}, \overline{BC}DE.$
$ABC, \overline{AB}D, A\overline{BCD}, A\overline{BDE}, A\overline{C}DE, \overline{ABC}\overline{E} \vdash ABDE, A\overline{BC}E, AC\overline{DE}, \overline{B}C\overline{DE}, \overline{BC}DE.$
$ABC, \overline{AB}D, ABDE, A\overline{BC}E, A\overline{BDE}, \overline{ABC}\overline{E} \vdash A\overline{BCD}, AC\overline{DE}, A\overline{C}DE, \overline{B}C\overline{DE}, \overline{BC}DE.$
$ABC, \overline{AB}D, ABDE, A\overline{BCD}, \overline{B}C\overline{DE}, \overline{BC}DE \vdash A\overline{BC}E, A\overline{BDE}, AC\overline{DE}, A\overline{C}DE, \overline{ABC}\overline{E}.$
$ABC, \overline{AB}D, ABDE, A\overline{BCD}, A\overline{BDE}, \overline{ABC}\overline{E}, \overline{BC}DE \vdash A\overline{BC}E, AC\overline{DE}, A\overline{C}DE, \overline{B}C\overline{DE}.$
$ABC, \overline{AB}D, A\overline{BCD}, A\overline{BC}E, A\overline{C}DE, \overline{B}C\overline{DE} \vdash ABDE, A\overline{BDE}, AC\overline{DE}, \overline{ABC}\overline{E}, \overline{BC}DE.$
$ABC, \overline{AB}D, ABDE, A\overline{BCD}, AC\overline{DE}, \overline{ABC}\overline{E}, \overline{BC}DE \vdash A\overline{BC}E, A\overline{BDE}, A\overline{C}DE, \overline{B}C\overline{DE}.$
48. $ABC, \overline{AB}D, A\overline{BC}E, A\overline{C}DE, \overline{A}B\overline{DE}, \overline{A}C\overline{DE} \vdash ABDE, \overline{ABC}\overline{E}, BC\overline{DE}, \overline{BC}DE.$
$ABC, \overline{AB}D, A\overline{BC}E, A\overline{C}DE, \overline{A}B\overline{DE}, \overline{ABC}\overline{E} \vdash ABDE, \overline{A}C\overline{DE}, BC\overline{DE}, \overline{BC}DE.$
$ABC, \overline{AB}D, ABDE, A\overline{BC}E, \overline{A}B\overline{DE}, \overline{ABC}\overline{E} \vdash A\overline{C}DE, \overline{A}C\overline{DE}, BC\overline{DE}, \overline{BC}DE.$
49. $AB, \overline{A}CE, \overline{B}CD \vdash ACD, BCE, CDE.$
50. $AB, CDE, \overline{AB}D\overline{E}, \overline{B}C\overline{DE} \vdash AC\overline{DE}, \overline{AB}CD, \overline{ABC}\overline{E}.$

$AB, CDE, AC\overline{DE}, \overline{AB}C\overline{E}, \overline{AB}D\overline{E} \vdash \overline{AB}CD, \overline{B}C\overline{DE}.$

51. $AB, \overline{AB}C, \overline{B}C\overline{DE}, \overline{BC}DE \vdash AC\overline{DE}, A\overline{C}DE, \overline{AB}DE.$
$AB, \overline{AB}C, A\overline{C}DE, \overline{AB}DE, \overline{B}C\overline{DE} \vdash AC\overline{DE}, \overline{BC}DE.$
$AB, \overline{AB}C, AC\overline{DE}, \overline{BC}DE \vdash A\overline{C}DE, \overline{AB}DE, \overline{B}C\overline{DE}.$
$AB, \overline{AB}C, AC\overline{DE}, A\overline{C}DE, \overline{AB}DE \vdash \overline{B}C\overline{DE}, \overline{BC}DE.$

52. $AB, \overline{AB}C, \overline{AC}DE, \overline{B}C\overline{DE} \vdash AC\overline{DE}, \overline{AB}DE, B\overline{C}DE.$
$AB, \overline{AB}C, AC\overline{DE}, \overline{AC}DE \vdash \overline{AB}DE, B\overline{C}DE, \overline{B}C\overline{DE}.$
$AB, \overline{AB}C, \overline{AB}DE, B\overline{C}DE, \overline{B}C\overline{DE} \vdash AC\overline{DE}, \overline{AC}DE.$
$AB, \overline{AB}C, AC\overline{DE}, \overline{AB}DE, B\overline{C}DE \vdash \overline{AC}DE, \overline{B}C\overline{DE}.$

53. $ABE, ACE, AD\overline{E}, BC\overline{E}, BDE \vdash ABC, ABD, ACD, BCD.$

54. $AB, BDE, \overline{B}CD, \overline{B}CE \vdash ACD, ACE, CDE.$

55. $AB, CDE, \overline{AB}C\overline{E}, \overline{AC}D\overline{E}, \overline{B}C\overline{DE} \vdash AC\overline{DE}, \overline{AB}CD, \overline{AB}D\overline{E}, B\overline{C}D\overline{E}.$
$AB, CDE, \overline{AB}CD, \overline{AC}D\overline{E}, \overline{B}C\overline{DE} \vdash AC\overline{DE}, \overline{AB}C\overline{E}, \overline{AB}D\overline{E}, B\overline{C}D\overline{E}.$
$AB, CDE, AC\overline{DE}, \overline{AB}C\overline{E}, \overline{AC}D\overline{E} \vdash \overline{AB}CD, \overline{AB}D\overline{E}, B\overline{C}D\overline{E}, \overline{B}C\overline{DE}.$
$AB, CDE, AC\overline{DE}, \overline{AB}C\overline{E}, \overline{AB}D\overline{E}, B\overline{C}D\overline{E} \vdash \overline{AB}CD, \overline{AC}D\overline{E}, \overline{B}C\overline{DE}.$

56. $AB, \overline{AB}C, \overline{C}DE, \overline{B}C\overline{DE} \vdash AC\overline{DE}, \overline{AB}DE.$
$AB, \overline{AB}C, \overline{C}DE, AC\overline{DE} \vdash \overline{AB}DE, \overline{B}C\overline{DE}.$

57. $ABE, ACE, AD\overline{E}, BC\overline{E}, BDE, CDE \vdash ABC, ABD, ACD, BCD.$

58. $AB, \overline{B}CE, \overline{B}DE, CD\overline{E} \vdash ACD, ACE, ADE, \overline{B}CD.$

59. $AB, \overline{A}DE, \overline{B}CD, \overline{B}CE \vdash ACD, ACE, BDE, CDE.$

60. $AC, BD, \overline{ACD}E, \overline{BCD}E \vdash A\overline{BD}E, \overline{A}B\overline{C}E.$
$AC, BD, \overline{A}B\overline{C}E, \overline{BCD}E \vdash A\overline{BD}E, \overline{ACD}E.$

61. $AB, \overline{A}CE, \overline{B}CD, \overline{BCDE} \vdash ACD, BCE, CDE, A\overline{CDE}.$

62. $AB, CDE, \overline{AB}D\overline{E}, \overline{AC}D\overline{E}, \overline{B}C\overline{DE} \vdash AC\overline{DE}, \overline{AB}CD, \overline{AB}C\overline{E}, B\overline{CDE}.$
$AB, CDE, AC\overline{DE}, \overline{AB}C\overline{E}, \overline{AB}D\overline{E}, \overline{ACD}E \vdash \overline{AB}CD, B\overline{CD}E, \overline{B}C\overline{DE}.$

63. $AC, BD, \overline{CD}E \vdash ABE, A\overline{D}E, B\overline{C}E.$

64. $AB, BDE, \overline{B}CD, \overline{B}CE, \overline{ACDE} \vdash ACD, ACE, CDE, B\overline{CDE}.$

65. $AC, BD, \overline{ABCD}, \overline{ACD}E, \overline{BCD}E \vdash A\overline{BD}E, \overline{A}B\overline{C}E.$
$AC, BD, \overline{A}B\overline{C}E, \overline{ABCD}, \overline{BCD}E \vdash A\overline{BD}E, \overline{ACD}E.$
$AC, BD, A\overline{BD}E, \overline{A}B\overline{C}E, \overline{ABCD} \vdash \overline{ACD}E, \overline{BCD}E.$

66. $AC, BD, \overline{ACD}E, \overline{BCD}E \vdash A\overline{BD}E, \overline{A}B\overline{C}E, \overline{ABCD}.$
$AC, BD, \overline{A}B\overline{C}E, \overline{ABCD}, \overline{BCD}E \vdash A\overline{BD}E, \overline{ACD}E.$
$AC, BD, A\overline{BD}E, \overline{A}B\overline{C}E, \overline{ABCD} \vdash \overline{ACD}E, \overline{BCD}E.$

67. $AB, AC, \overline{A}DE, \overline{BDE} \vdash A\overline{DE}, BDE, CDE.$

68. $AC, B\overline{C}, \overline{B}DE \vdash AB, ADE, \overline{C}DE.$

69. $AB, \overline{B}CE, \overline{B}DE, CD\overline{E}, \overline{ACDE} \vdash ACD, ACE, ADE, \overline{B}CD, B\overline{CDE}.$

70. $AB, \overline{A}DE, \overline{B}CD, \overline{B}CE, \overline{ACDE} \vdash ACD, ACE, BDE, CDE, B\overline{CDE}.$

71. $AC, B\overline{C}, C\overline{DE}, \overline{C}DE \vdash AB, ADE, B\overline{DE}.$

72. $AB, AC, \overline{ADE}, \overline{B}DE, \overline{C}DE \vdash ADE, B\overline{DE}, C\overline{DE}.$

73. $AC, B\overline{C}, \overline{A}DE, \overline{CDE} \vdash AB, A\overline{DE}, BDE, CDE.$

74. $AC, B\overline{C}, \overline{ADE}, \overline{B}DE \vdash AB, ADE, B\overline{DE}, C\overline{DE}, \overline{C}DE.$

75. $A\overline{C}, AE, BC, B\overline{E}, CD \vdash AB, AD.$

76. $A\overline{D}, AE, B\overline{C}, B\overline{E}, CD \vdash AB, AC, BD.$

77. $A\overline{D}, A\overline{E}, BD, BE, CD, CE \vdash AB, AC.$

78. $A\overline{B}, BD, BE, C\overline{D} \vdash AC, AD, AE, BC.$

79. $AE, B\overline{E}, C\overline{E}, DE \vdash AB, AC, BD, CD.$

80. $AD, AE, BD, B\overline{E}, C\overline{D}, CE \vdash AB, AC, BC.$

81. $A\overline{D}, A\overline{E}, B\overline{C}, CD, CE \vdash AB, AC, BD, BE.$

82. $A\overline{B}, BE, C\overline{E}, D\overline{E} \vdash AC, AD, AE, BC, BD.$

83. $AD, AE, B\overline{E}, C\overline{D}, DE \vdash AB, AC, BC, BD, CE.$

84. $A\overline{D}, B\overline{C}, C\overline{E}, DE \vdash AB, AC, AE, BD, B\overline{E}, CD.$

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