Relational Logic and Arithmetic

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The arithmetic is a science about the numbers and operations with the numbers. This science consists of two main parts - the numerical models and the number theory.

In the relational logic all models are numerical. All models are constructed from the set of the natural numbers N by the operations of power set and direct product.

The numerical models are stated in detail in [1]. In the given paper we state only some problems of the number theory.

We study the numbers received by the fast increasing function and its inverse functions. The first argument all of these functions defines the order of numbers. The natural numbers have order 0, the integers - order 1, the rational numbers - order 2. There exist numbers of the third and higher orders.

There are other numbers such as real and infinitesimal, but we do not search them here.

1. Arithmetic of Natural Numbers.

This arithmetic is a theory with one sort of variables N and with one relational symbol " $'$ " (there is no function or constant symbols in relational logic). The sort is interpreted by the set of the natural numbers and the relation is interpreted by the succession (addition of 1). The variables of the natural numbers are denoted by N_0, N_1, N_2,... .

The relation of succession exists and one-valued. This statement is a valid formula in logic with induction. So we can replace the relation $'(N_1, N_2)$ by equality $N_1' = N_2$ and can use N_1' as a term.

The first axiom of arithmetic states the successor function is injective:

$$N_1' \neq N_2' \vee N_1 = N_2 \tag{Ar1}$$

Further we should define the relation 0:

$$0(N_1) \rightleftarrows \forall N_2 \ N_2' \neq N_1 \text{ - there exist no successor equal to } 0$$

The relation $0(N_1)$ exists. This is a valid formula in logic with induction. As it follows from below, $0(N_1)$ is one-valued. So $0(N_1)$ is a constant. We can denote $0(N_1)$ as $0 = N_1$ and 0 can be used as a term.

Therefore, the definition of 0 can be presented as

$$0 = N_1 \rightleftarrows \forall N_2 \ N_2' \neq N_1$$

After reducing this definition to the first normal form (NF1, [1]) we have

$$0 = x_1 \vee x_1 = a_1' \tag{1}$$

$$x_1' \neq 0 \tag{2}$$

where x_1 is a variable, bound by universal quantifier, a_1 is a variable, bound by existential quantifier.

The first of these formulas is a positive definition [1], the second one is a negative definition.

The positive definition (2) is superfluous because it is deduced from (Ar1). We will prove this by reduction ad absurdum.

Negation of the positive definition is $0 \neq c_1 \wedge c_1 \neq x_1'$. After adding $c_1' \neq x_1'$ and applying the finite descent rule we receive the inconsistent literal $0 \neq 0$.

But we should show negation of the added literal is inconsistent too.

The resolution of the literal $c_1' = x_1'$ and (Ar1) gives $c_1 = x_1$. The quantifiers $\exists c_1 \forall x_1$ present in the latter formula implicitly and this means $c_1 = x_1$ is an inconsistent literal. The proof is complete.

Therefore, the positive definition of 0 can be removed from the axiom system. After returning to previous notation we have the negative definition as the second axiom of arithmetic:

$$N'_1 \neq 0 \tag{Ar2}$$

Now we should define the order in arithmetic:

$$N_1 \leqslant N'_1 \tag{Ar3}$$

$$N'_1 \leqslant N_2 \vee N_2 \leqslant N_1 \tag{Ar4}$$

Formula (Ar3) introduces an order on N. Formula (Ar4) removes all the natural numbers between N_1 and N'_1 because for every N_1 and N_2 it should be $N_2 \geqslant N'_1$ or $N_2 \leqslant N_1$.

Theorem 1.1. *It is follows from* (Ar3), (Ar4), *and from transitivity of an order that the introduced order is linear:*

$$N_1 \leqslant N_2 \vee N_2 \leqslant N_1.$$

Proof:

1. $x_1 \leqslant x'_1$ - formula (Ar3).
2. $x_1 \leqslant x_2 \vee x'_2 \leqslant x_1$ - formula (Ar4).
3. $c_1 \nleqslant c_2 \wedge c_2 \nleqslant c_1$ - negation of the theorem.
4. $(2,3)\breve{}c'_2 \leqslant c_1 \wedge c_1 \nleqslant c_2 \wedge \breve{}c_2 \nleqslant c_1$.

In brackets we give the numbers of sentences used in the resolution rule.

We underline the literals used in resolutions of sentence 4 and transitivity of the order:

$$x_1 \nleqslant x_2 \vee \breve{}x_2 \nleqslant x_3 \vee \breve{}x_1 \leqslant x_3$$

So we have as the result of resolution:

4$'$. $c_2 \nleqslant c'_2 \wedge c'_2 \leqslant c_1 \wedge c_1 \nleqslant c_2 \wedge c_2 \nleqslant c_1$.

5. (1,4$'$) empty. □

Let us mark the order transitivity is a valid formula in logic with order. So this formula misses among source formulas and is used only for transforming other formulas.

As we say above there is no natural number between N_1 and N'_1. So arithmetic of the natural numbers has the unique model within isomorphism if every natural number is finite. But every natural number is finite because this statement is a valid formula in logic with induction.

By virtue of the unique model, 0 is unique too. This was used above.

2. Fast Increasing Function.

The fast increasing function is denoted by $P(N_0, N_1, N_2)$ and has next properties:

$$P(0, N_1, N_2) = N'_2 \tag{2.1}$$

$$P(1, N_1, 0) = N_1 \tag{2.2}$$

$$P(2, N_1, 0) = 0 \tag{2.3}$$

$$P(3, 0, 0) = N_3 \tag{2.4}$$

$$P(3, N'_1, 0) = 1 \tag{2.5}$$

$$P(N''''_0, N_1, 0) = 1 \tag{2.6}$$

$$P(N'_0, N_1, N'_2) = P(N_0, N_1, P(N'_0, N_1, N_2)) \tag{2.7}$$

The function $P(N_0, N_1, N_2)$ generalizes the basic arithmetical functions. They are succession ($N_0 = 0$), addition ($N_0 = 1$), multiplication ($N_0 = 2$), exponential ($N_0 = 3$) etc. If the successor function is given, all other functions are defined through it.

In particular, the classic definition of the operation of addition follows from the formulas (2.2) and (2.7):

$$N_1 + 0 = N_1.$$

$$N_1 + N'_2 = (N_1 + N_2)'.$$

Let us show, that in the relational logic we shall receive the same definition of this operation.

The operation of addition in relational logic is defined by the formulas:

$$+(N_1, N_2, N_3) \rightleftarrows \exists N_4 \exists N_5 \; N'_4 = N_1 \wedge N'_5 = N_3 \wedge +(N_1, N_4, N_5).$$

$$+(N_1, 0, N_2) \rightleftarrows N_1 = N_2.$$

As it follows from this definition, the relation of addition exists and is one-valued. It means we can use the relation $N_1 + N_2 = N_3$ and the term $N_1 + N_2$.

Replacing equivalence by implications we shall receive positive and negative definitions:

$$N_1 + N_2 = N_3 \vee N'_4 \neq N_2 \vee N'_5 \neq N_3 \vee N_1 + N_4 \neq N_5.$$

$$N_1 + N_2 \neq N_3 \vee \exists N_4 \exists N_5\ N_4' = N_2 \wedge N_5' = N_3 \wedge N_1 + N_4 = N_5.$$
$$N_1 + 0 = N_2 \vee N_1 \neq N_2.$$
$$N_1 + 0 \neq N_2 \vee N_1 = N_2.$$

Using properties of equality we can simplify these sentences:

$$N_1 + N_4' = (N_1 + N_4)'.$$
$$\exists N_4\ N_1 + N_4' = (N_1 + N_4)'.$$
$$N_1 + 0 = N_1.$$
$$N_1 + 0 = N_1.$$

The second formula is a special case of first and can be omitted. The fourth formula coincides with third and can be omitted too.

After changing variables in the first formula we have the classic definition of addition.

This result is true for all primitive recursive functions.

The operation of multiplication is defined by formulas (2.3) and (2.7):

$$N_1 \cdot 0 = 0$$
$$N_1 \cdot N_2' = N_1 \cdot N_2 + N_1.$$

We can deduce the properties of commutability and associativity for addition and multiplication and their distributivity with the help of the finite descent rule.

The exponential operation is defined by the formulas (2.4), (2.5) and (2.7):

$$0^0 = N_1.$$
$$N_1'^{\,0} = 1.$$
$$N_1^{N_2'} = N_1^{N_2} \cdot N_1.$$

As it follows from the first formula, exponential is multi-valued at point $N_1 = 0, N_2 = 0$.

The other operations are defined by formulas (2.6) and (2.7). The next formula is obtained by induction:

$$P(N_0'', N_1, 1) = N_1 \tag{2.8}$$

The fast increasing function is needed to construct the natural, integer, rational and other numbers.

At constructing we use the uniform schema.

Let us define inverse functions of $P(N_0, N_1, N_2) = N_3$ at a given N_0. Such functions are two:

$$P^1(N_0, N_1, N_2) = N_3 \rightleftarrows P(N_0, N_3, N_2) = N_1$$
$$P^2(N_0, N_1, N_2) = N_3 \rightleftarrows P(N_0, N_1, N_3) = N_2$$

We extend functions P^1 and P^2 if we replace N_3 by $< N_1, N_2 >$. Thus, for each function $P(N_0, N_1, N_2) = N_3$ two inverse functions are defined:

$$P^1(N_0, N_1, N_2) =< N_1, N_2 >$$
$$P^2(N_0, N_1, N_2) =< N_1, N_2 >$$

Now the inverse functions are defined for all values N_1 and N_2, but there are sets of equal pairs and we must select one representative for each of these sets.

Let us have the set of equal pairs $\{< N_1.N_2 >, < N_3, N_4 >, < N_5, N_6 >, ...\}$.

We can select the pair $< N_i, N_j >$ with the minimal value of N_i or with the minimal value of N_j as a single representative.

But we select the pair with the minimal value of N_i for function P^1, and the pair with the minimal value of N_j for function P^2.

Such choice is done because the first element of the pair in function P^1 is a value of function P. For function P^2, the second element of the pair is a value of function P.

So we have two sorts of numbers. The first sort is a set of pairs $< N_1, N_2 >$ with the minimal value of N_1 among equal pairs for function P^1. The second sort is a set of pairs $< N_1, N_2 >$ with the minimal value of N_2 among equal pairs for function P^2.

Further both sorts are joined and extended to be closed by all arithmetic operations of order N_0 and less.

On this the construction of the new sort of the numbers is finished.

3. Arithmetic of Integer Numbers.

In the formulas (2.1)-(2.7) we suppose, that the natural numbers are already constructed. This supposition is true (see section 1).

Now we can use the formulas (2.2) and (2.7) for construction of integers.

For this we use the addition function $P(1, N_1, N_2)$ and its inverse function.

The inverse function of addition is subtraction:

$$N_1 - N_2 = N_3 \rightleftarrows N_3 + N_2 = N_1$$

Let us have a new sort $< N_1, N_2 >$:

$$N_1 - N_2 =< N_1, N_2 >$$

Such pair is ambiguous, since:

$$< N_1, N_2 >=< N_1 + N_3, N_2 + N_3 >$$

$$N_3 \leqslant N_1 \wedge N_3 \leqslant N_2 \rightarrow < N_1, N_2 >=< N_1 - N_3, N_2 - N_3 >$$

We must find the minimal value of the first element of the pair in the right part of these equalities.

For that we can use only the second equality. Then the first element of the pair is minimal at $N_3 = N_2$, if $N_1 \geqslant N_2$, and at $N_3 = N_1$, if $N_1 \leqslant N_2$.

So the definition of the sort of integers is finished: $< N_1 - N_2,\ 0 >$ for the positive numbers, and $< 0, N_2 - N_1 >$ for the negative numbers. The integer 0 is $< 0, 0 >$ and is neither positive nor negative.

Let Z be a sort of integers. Then

$$Z(N_1, N_2,) \rightleftarrows N_1 = 0 \vee N_2 = 0 \tag{Z1}$$

In this definition one argument is anonymous. This argument is last in the relation and the value of this argument is an object of this sort. This allows to formulate the axioms for arithmetic of integers.

The definition of zero is an axiom:

$$Z(0, 0, 0) \tag{Z2}$$

The zero as natural or integer number has the same designation. This ambiguity of the designation is admissible if the rules of the type reduction exist (the types are copies of a sort, one sort can have several types). These rules are similar to ones existed in computer languages.

We can define both the positive and negative numbers:

$$Z(N_1, 0, N_1) \tag{Z3}$$

$$Z(0, N_1, -N_1) \tag{Z4}$$

The order of integers is put up by the axiom:

$$\begin{gathered} Z_1 \leqslant Z_2 \rightleftarrows \exists N_1 \exists N_2\ Z(N_1, 0, Z_1) \wedge Z(N_2, 0, Z_2) \wedge N_1 \leqslant N_2 \quad \vee \\ Z(0, N_1, Z_1) \wedge Z(0, N_2, Z_2) \wedge N_2 \leqslant N_1 \quad \vee \quad Z(0, N_1, Z_1) \wedge Z(N_2, 0, Z_2) \end{gathered} \tag{Z5}$$

It follows from this axiom, the order is linear:

$$Z_1 \leqslant Z_2 \vee Z_2 \leqslant Z_1.$$

The addition of integers is defined by the axiom:

$$\begin{gathered} Z_1 + Z_2 = Z_3 \rightleftarrows \exists N_1 \exists N_2\ Z(N_1, 0, Z_1) \wedge Z(N_2, 0, Z_2) \wedge Z(N_1 + N_2, 0, Z_3) \quad \vee \\ Z(0, N_1, Z_1) \wedge Z(0, N_2, Z_2) \wedge Z(0, N_1 + N_2, Z_3) \quad \vee \\ Z(N_1, 0, Z_1) \wedge Z(0, N_2, Z_2) \wedge N_1 > N_2 \wedge Z(N_1 - N_2, 0, Z_3) \quad \vee \\ Z(N_1, 0, Z_1) \wedge Z(0, N_2, Z_2) \wedge N_1 \leqslant N_2 \wedge Z(0, N_2 - N_1, Z_3) \quad \vee \\ Z(0, N_1, Z_1) \wedge Z(N_2, 0, Z_2) \wedge N_1 > N_2 \wedge Z(0, N_1 - N_2, Z_3) \quad \vee \\ Z(0, N_1, Z_1) \wedge Z(N_2, 0, Z_2) \wedge N_1 \leqslant N_2 \wedge Z(N_2 - N_1, 0, Z_3) \end{gathered} \tag{Z6}$$

The change of the sign of an integer is defined by the formula:

$$-Z_1 = Z_2 \rightleftarrows \exists N_1\ Z(N_1, 0, Z_1) \wedge Z(0, N_1, Z_2) \quad \vee \quad Z(0, N_1, Z_1) \wedge Z(N_1, 0, Z_2) \tag{Z7}$$

Let's define an absolute value of an integer:

$$|Z_1| = Z_2 \rightleftarrows \exists N_1\ Z(N_1, 0, Z_1) \wedge Z(N_1, 0, Z_2) \quad \vee \quad Z(0, N_1, Z_1) \wedge Z(N_1, 0, Z_2) \tag{Z8}$$

The operation of substruction is defined very simple:

$$Z_1 - Z_2 = Z_1 + (-Z_2) \tag{Z9}$$

The definition of multiplication is last:

$$\begin{gathered} Z_1 \cdot Z_2 = Z_3 \rightleftarrows \exists N_1 \exists N_2\ Z(N_1, 0, Z_1) \wedge Z(N_2, 0, Z_2) \wedge Z(N_1 \cdot N_2, 0, Z_3) \quad \vee \\ Z(0, N_1, Z_1) \wedge Z(0, N_2, Z_2) \wedge Z(N_1 \cdot N_2, 0, Z_3) \quad \vee \\ Z(N_1, 0, Z_1) \wedge Z(0, N_2, Z_2) \wedge Z(0, N_1 \cdot N_2, Z_3) \quad \vee \\ Z(0, N_1, Z_1) \wedge Z(N_2, 0, Z_2) \wedge Z(0, N_1 \cdot N_2, Z_3) \end{gathered} \tag{Z10}$$

The construction of arithmetic of integers is finished.

4. Arithmetic of Rational Numbers.

The rational numbers appear at construction of the inverse function for multiplication.

The inverse function for multiplication is division:

$$N_1/N_2 = N_3 \rightleftarrows N_3 \cdot N_2 = N_1$$

Let us enter a new sort of the numbers $< N_1, N_2 >$:

$$N_1/N_2 =< N_1, N_2 >$$

There is a set of equal pairs, since

$$\forall N_3 \neq 0 \;\; < N_1, N_2 >=< N_1 \cdot N_3, N_2 \cdot N_3 >$$

$$\forall N_3 \neq 0 \;\; N_3 \in \mathrm{div}(N_1) \wedge N_3 \in \mathrm{div}(N_2) \rightarrow < N_1, N_2 >=< N_1/N_3, N_2/N_3 >$$

where $\mathrm{div}(N_0)$ means divisors of N_0. The first element of the pair is minimal, if N_3 is the greatest common devisor (GCD) of N_1 and N_2: $N_3 = \mathrm{GCD}(N_1, N_2)$. After we divide N_1 and N_2 on their GCD, the greatest common devisor for new values N_1 and N_2 becomes equal to 1.

So we construct the new sort as the extension of the natural numbers. Let this sort be denote by Q^+. Then we have the definition:

$$Q^+(N_1, N_2,) \rightleftarrows N_1 = 0 \wedge N_2 = 1 \quad \vee \quad N_1 \neq 0 \wedge N_2 \neq 0 \wedge \mathrm{GCD}(N_1, N_2) = 1.$$

This sort is a set of non-negative rational numbers, the numerator is N_1, the denominator is N_2.

But we must construct the new sort as the extension of the integer numbers. This is not difficult.

Let the new sort be noted by Q. The definition of the sort is:

$$Q(Z_1, N_1,) \rightleftarrows Z_1 = 0 \wedge N_1 = 1 \quad \vee \quad Z_1 \neq 0 \wedge N_1 \neq 0 \wedge \mathrm{GCD}(|Z_1|, N_1) = 1 \tag{Q1}$$

This sort is a set of all rational numbers, the numerator is Z_1, the denominator is N_1.

Now we can give definition of the rational number 0:

$$Q(0, 1, 0) \tag{Q2}$$

The order of rational numbers is put up by the formula:

$$Q_1 \leqslant Q_2 \rightleftarrows \exists Z_1, Z_2, N_1, N_2 \;\; Q(Z_1, N_1, Q_1) \wedge Q(Z_2, N_2, Q_2) \wedge Z_1 \cdot N_2 \leqslant Z_2 \cdot N_1 \tag{Q3}$$

The addition, substruction and multiplication of the rational numbers are similarly defined. The division is defined a bit more complex:

$$Q_1/Q_2 = Q_3 \rightleftarrows Q_1 = 0 \wedge Q_2 = 0 \quad \vee \quad Q_1 = 0 \wedge Q_2 \neq 0 \wedge Q_3 = 0 \quad \vee \tag{Q4}$$

$$Q_1 \neq 0 \wedge Q_2 \neq 0 \wedge \exists Z_1, Z_2, Z_3, N_1, N_2, N_3 \;\; Q(Z_1, N_1, Q_1) \wedge Q(Z_2, N_2, Q_2) \wedge Q(Z_3, N_3, Q_3) \wedge$$

$$Z_3 = \frac{Z_1}{\mathrm{GCD}(|Z_1|, |Z_2|)} \cdot \frac{Z_2}{|Z_2|} \cdot \frac{N_2}{\mathrm{GCD}(N_1, N_2)} \wedge N_3 = \frac{Z_1}{|Z_1|} \cdot \frac{Z_2}{\mathrm{GCD}(|Z_1|, |Z_2|)} \cdot \frac{N_1}{\mathrm{GCD}(N_1, N_2)}$$

So 0/0 is equal to every number, in other cases division on 0 is not equal to any number.

5. Arithmetic of High Order Numbers.

The arithmetical numbers of the first and second orders are integer and rational numbers.

The arithmetical numbers of the third order are constructed by inversion of the fast increasing function $P(3, N_1, N_2)$. This function is exponential.

The first inverse function of exponential is rooting:

$$\sqrt[N_2]{N_1} = N_3 \rightleftarrows N_3^{N_2} = N_1$$

In common case rooting is a many-valued function, but we take the main branch of it.

At $N_1 = 1, N_2 = 0$ the function does not exist because N_3 is fictive. At N_1=0 we have fictive N_2, at $N_2 = 0$ we have fictive N_1, but the function exists. At $N_2 = 1$ we have $N_3 = N_1$, i.e. the values of the function are all natural numbers. The other values exist at $N_1 > 1, N_2 > 1$.

Let us enter the new sort $< N_1, N_2 >$:

$$\sqrt[N_2]{N_1} =< N_1, N_2 >$$

We choice $< N_1, 1 >$ if the value of rooting is a natural number. The pairs $< N_1, N_2 >$ have $N_1 \neq N_0^{N_2}$ (for all N_0) if rooting has irrational values. In this case we have a set of equal pairs:

$$\forall N_0 \neq 0 \;\; < N_1, N_2 >=< N_1^{N_0}, N_2 \cdot N_0 >$$

$$\forall N_0 \neq 0 \;\; < N_1, N_2 >=< \sqrt[N_0]{N_1}, N_2/N_0 >$$

We must take a representative with the minimal value of the first element of equal pairs.

Only the last formula can be used for this. The formula defines the representative with the maximal N_0.

Such representatives are defined by the equation:

$$\mathrm{GCD}(\mathrm{expm}(N_1), N_2) = 1 \tag{1}$$

where $\mathrm{expm}(N_1)$ is the maximal exponent for $N_1 > 1$, i.e. maximal N_3 for all N_4 such that $N_1 = N_4^{N_3}$. The other exponents are divisors of the maximal N_3.

The equation (1) is true, because the maximal N_0 in $<\sqrt[N_0]{N_1}, N_2/N_0>$ equals $\mathrm{GCD}(\mathrm{expm}(N_1), N_2)$. Using this N_0 we get a new pair $< N_1^*, N_2^* >$, where $N_1^* = \sqrt[N_0]{N_1}$, $N_2^* = N_2/N_0$. For this new pair we have $\mathrm{GCD}(\mathrm{expm}(N_1^*), N_2^*) = 1$.

Thus, the new sort is constructed as the extension of the set of natural numbers. Let this sort be denoted by r^+. Then we have the definition:

$$r^+(N_1, N_2,) \rightleftarrows N_2 = 1 \ \vee \ \mathrm{GCD}(\mathrm{expm}(N_1), N_2) = 1 \wedge \forall N_0 \ N_1 \neq N_0^{N_2}$$

But we must also construct the new sort as the extension of the rational number set. This extension is simple.

Let the new sort be denoted by r. Then the definition is

$$r(Q_1, Q_2,) \rightleftarrows Q_2 = 1 \quad \vee \tag{r1}$$

$$Q_2 > 0 \wedge \mathrm{GCD}(\mathrm{expm}(|Q_1|), \mathrm{num}(Q_2)) = 1 \wedge (Q_1 > 0 \vee \mathrm{odd}(\mathrm{den}(Q_2))) \wedge \forall Q_0 \ Q_1 \neq Q_0^{Q_2}$$

where the functions "num" and "den" give a numerator and denominator of rational numbers, the relation "odd" picks out the odd natural numbers, and the function "expm" gives the maximal natural exponent of rational numbers.

We have positive rooting if $Q_1 > 0$. We have negative rooting if $Q_1 < 0$, $\mathrm{num}(Q_2)$ is odd, and $\mathrm{den}(Q_2)$ is odd too. We have rooting with the complex factor $\sqrt[2n]{-1}$ $(n \geqslant 1)$ if $Q_1 < 0$, $\mathrm{num}(Q_2)$ is even, and $\mathrm{den}(Q_2)$ is odd.

The definition of a new sort r is completed. It is the sort of radical numbers. But we have simultaneously the extension of exponential numbers if we replace Q_2 by $1/Q_2$ in the right part of this definition.

Let us define constants 0, ±1 and the imaginary unity:

$$\mathrm{r}(0, 1, 0) \tag{r2}$$

$$\mathrm{r}(1, 1, 1) \tag{r3}$$

$$\mathrm{r}(-1, 1, -1) \tag{r4}$$

$$\mathrm{r}(-1, 2, i) \tag{r5}$$

It is impossible to define the basic operations of arithmetic. In particular, the sum of radical numbers is not a radical number as a rule.

But r can be get from r^+ by using the arithmetic operations:

$$< Q_1, Q_2 >= \frac{< \mathrm{num}(|Q_1|), \mathrm{num}(|Q_2|) > \cdot \ \mathrm{num}(|Q_1|)^{\mathrm{den}(Q_2)}}{< \mathrm{den}(Q_1), \mathrm{num}(|Q_2|) > \cdot \ \mathrm{den}(Q_1)^{\mathrm{den}(Q_2)}}$$

Further we construct a sort of numbers from the second inverse function.

The second inverse function of exponential is logarithm:

$$\log_{N_2} N_1 = N_3 \rightleftarrows N_2^{N_3} = N_1$$

We take the main branch of logarithm. At $N_1 = N_2 = 1$ logarithm does not exist because N_3 becomes fictive. At $N_2{=}0$ we have fictive N_1, at $N_1 = 1$ we have fictive N_2, but logarithm exists $(N_3 = 0)$. At $N_1 = N_4^{N_5}, N_2 = N_4^{N_6}$ logarithm has rational values. In the other cases logarithm has irrational values.

Let the new sort be $< N_1, N_2 >$:

$$\log_{N_2} N_1 =< N_1, N_2 >$$

We take the representative $< 2^{N_3}, 2^{N_4} >$ if $\mathrm{GCD}(N_3, N_4) = 1$ and logarithm has rational values. The pairs $< N_1, N_2 >$ have $N_1 \neq N_2^{N_0}$ if logarithm has irrational values.

Let the new sort be denoted by l^+. Then we have the definition:

$$\mathrm{l}^+(N_1, N_2,) \rightleftarrows (\exists N_3 \exists N_4 {\neq} 0 \ \mathrm{GCD}(N_3, N_4) = 1 \wedge N_1 = 2^{N_3} \wedge N_2 = 2^{N_4}) \quad \vee$$

$$N_1 > 1 \wedge N_2 > 1 \wedge \mathrm{GCD}(\mathrm{expm}(N_1), \mathrm{expm}(N_2)) = 1 \wedge \forall N_0 \ N_1 \neq N_2^{N_0}$$

But we should construct the new sort as the extension of the set of rational numbers too. Let this extension be denoted by l:

$$\mathrm{l}(Q_1, Q_2,) \rightleftarrows \mathrm{GCD}(|Q_1|, |Q_2|) = 1 \wedge \exists Q_3 \ Q_1 = 2^{Q_3} \wedge Q_2 = 2 \quad \vee \tag{l1}$$

$$Q_1 > 0 \wedge Q_2 > 0 \wedge \forall Q_0 \ Q_1 \neq Q_2^{Q_0}$$

The definition of the new sort l is completed. It is the sort of logarithmic numbers.

We can define constants 0 and ±1:

$$\mathrm{l}(1,2,0) \tag{12}$$

$$\mathrm{l}(2,2,1) \tag{13}$$

$$\mathrm{l}(2^{-1},2,-1) \tag{14}$$

Sort l is not closed with respect to the operations of arithmetic. But l can be get from l^+ by using these operations:

$$< Q_1, Q_2 >= \frac{< \mathrm{num}(|Q_1|), 2 > - < \mathrm{den}(Q_1), 2 >}{< \mathrm{num}(|Q_2|), 2 > - < \mathrm{den}(Q_2), 2 >}$$

Both sorts r+ and l+ can be closed with respect to the arithmetic operations if we extend them to the complex numbers. This extension is the sort of the third order numbers.

Let the third order numbers be denote by P^3. The sort of the complex numbers is denoted by C, variables of this sort are denoted by C_0, C_1, C_2,... .

Definition 5.1. $P^3(C_0) \rightleftarrows (\exists N_0\ C_0 = N_0) \quad \vee \quad \exists C_1 \exists C_2\ P^3(C_1) \wedge P^3(C_2) \wedge$
$(C_0 = C_1 + C_2 \vee C_0 = C_1 - C_2 \vee C_0 = C_1 \cdot C_2 \vee C_0 = C_1/C_2 \vee C_0 = C_1^{C_2} \vee C_0 = \log_{C_1} C_2)$

This definition is inductive. The third order numbers are:

- the natural numbers N;
- sum and difference of the third order numbers;
- product and division of the third order numbers;
- exponential and logarithm of the third order numbers.

Here we remove rooting, because $\sqrt[C_1]{C_2}$ equals $C_2^{C_1^{-1}}$. We take the main branch of exponential and logarithm.

But before we must define the sort of the complex numbers: the complex number C_0 is $R_1 + iR_2$, and the real number R_0 is a section of the rational number set.

So the third order numbers are constructed. They have complex and transcendental numbers.

We can define the third order numbers without the less order. For that we must delete in r^+ and l^+ rational values of rooting and logarithm and replace the natural numbers by r^+ and l^+ in the definition 5.1. Besides, we must immediately delete rational numbers generated at induction.

It is necessary to mark, construction of the fourth order numbers is incomparably complex than construction of the third order numbers. This is because we should use non-elementary functions to construct the fourth order numbers.

For the fourth order numbers we must define function $P(4, C_1, C_2)$. This function is analytic continuation of function $P(4, N_1, N_2)$. Then we define both inverse functions and construct all numbers from the natural numbers by basic arithmetic operations, exponential, logarithm, function $P(4, C_1, C_2)$ and its both inverse functions.

Similarly we construct the higher order numbers.

References.

[1]. M.A. Malkov. Introduction to relational logic. *Relational logic*, 2001, 1.