Axioms of Classic Set Theory (NBG)

M.A.Malkov
Russian Research Center for Artificial Intelligence.
E-mail: ma@malkov.msk.su

Contents

In NBG-theory [1] undefined notions are the sort X and the relation " $\in$ ".

The sort X is interpreted as a collection of classes, the classes are interpreted as a generalization of notion "set", "$\in$" is interpreted as the relation of membership. The variables of the sort X are denoted by $X_0, X_1, X_2...$. These variables are interpreted as arbitrary classes: class X_0, class X_1, class X_2,... .

Classes have one-place members or n-tuples as members. Classes with one-place members we shall call the one-place classes. Classes with n-tuples ($n > 1$) we shall call the multi-place classes.

The theory of one-place classes is a subtheory of NBG (i.e. the axioms of the one-place classes are a subset of the axioms of NBG). The extension of this theory up to NBG converts some one-place classes in multi-place classes. An example of such classes is the universal class.

At first we state axioms of the theory of one-place classes, then we state the other axioms of NBG. In last two sections we reduce these axioms to the first normal form [2].

If NBG-theory is inconsistent, then an subset of its axioms can be inconsistent.

The majority of the axioms states the existence of relations and functions. This is natural, since relations and functions can be used if their domains are known. For functions, their ranges should be also known. The axioms of existence put these fields.

All axioms are labeled as usual [1]. Besides, all axioms are numbered and are marked by character A. The definitions of some notions are given for simplification of the axioms. Some definitions are superfluous, but they are widely used. All definitions are numbered too and are marked by character D. In the first normal form these axioms and definitions are only numbered.

The axioms and definitions are introduced by means of the first order logic with equality. So we include symbols of negation, conjunction, disjunction, implication, equivalence, and both existential and universal quantifiers:

$$\neg,\ \wedge,\ \vee,\ \rightarrow,\ \rightleftarrows,\ \exists,\ \forall$$

These symbols are enumerated in the sequence of their priorities. The priorities decrease a number of brackets.

1. One-Place Classes.

The defined notions of the one-place class theory are equality " $=$ ", inclusion " $\subseteq$ " and set M.

Two classes are equal, if they consist of identical members:

$$X_1 = X_2 \rightleftarrows \forall X_3\ X_3 \in X_1 \rightleftarrows X_3 \in X_2 \qquad (D1)$$

The class X_1 is included in X_2 (is the subclass of X_2), if all members of X_1 belong to X_2:

$$X_1 \subseteq X_2 \rightleftarrows \forall X_3\ X_3 \in X_1 \rightarrow X_3 \in X_2 \qquad (D2)$$

The set is a member of a class:

$$M(X_1) \rightleftarrows \exists X_2\ X_1 \in X_2 \qquad (D3)$$

There exist classes that are not members of classes. Such classes are called the proper classes. Their members are sets.

There exists empty class (set) that does not contain members:

N $$\exists X_0\ M(X_0) \wedge \forall X_1\ M(X_1) \rightarrow X_1 \notin X_0 \qquad (A1)$$

It follows from (D1) and (A1) that the empty set is unique. Let it be denoted by 0:

$$0 = X_1 \rightleftarrows \forall X_2\ M(X_2) \rightarrow X_2 \notin X_1 \qquad (D4)$$

There exist classes (sets) containing two members:

P $\forall X_1 \forall X_2\ M(X_1) \wedge M(X_2) \rightarrow \exists X_0\ M(X_0) \wedge \forall X_3\ M(X_3) \rightarrow (X_3 \in X_0 \rightleftarrows X_3 = X_1 \vee X_3 = X_2)(A2)$

It can be show that this set is unique (for given X_1 and X_2). Let it be denoted by $\{X_1, X_2\}$:

$$\{X_1, X_2\} = X_3 \rightleftarrows \forall X_4\ M(X_4) \rightarrow (X_4 \in X_3 \rightleftarrows X_4 = X_1 \vee X_4 = X_2) \qquad (D5)$$

If $X_1 = X_2$, the set $\{X_1, X_2\}$ becomes with one member and is denoted by $\{X_1\}$:

$$\{X_1\} = \{X_1, X_1\} \qquad (D6)$$

There exists a class X_0 that is intersection of classes X_1 and X_2:

B2 $$\forall X_1 \forall X_2 \exists X_0 \forall X_3\ M(X_3) \rightarrow (X_3 \in X_0 \rightleftarrows X_3 \in X_1 \wedge X_3 \in X_2) \qquad (A3)$$

The class X_0 is unique (for given X_1 and X_2). Let it be denoted by:

$$X_1 \cap X_2 = X_3 \rightleftarrows \forall X_4\ M(X_4) \rightarrow (X_4 \in X_3 \rightleftarrows X_4 \in X_1 \wedge X_4 \in X_2) \qquad (D7)$$

There exist classes with an infinite number of members. These classes are complements of X_0 to classes X_1:

B3 $$\forall X_1 \exists X_0 \forall X_2\ M(X_2) \rightarrow (X_2 \in X_0 \rightleftarrows X_2 \notin X_1) \qquad (A4)$$

The complement (for given X_1) is unique. Let it be denoted:

$$\overline{X}_1 = X_2 \rightleftarrows \forall X_3\ M(X_3) \rightarrow (X_3 \in X_2 \rightleftarrows X_3 \notin X_1) \qquad (D8)$$

The complement to the empty class is the universal class denoted by V:

$$V = \overline{0} \qquad (D9)$$

The members of the universal class are all sets, including the empty set. Moreover, any class X_1 is a subclass of V.

For the join of classes X_1 and X_2 we introduce the next definition:

$$X_1 \cup X_2 = \overline{\overline{X}_1 \cap \overline{X}_2} \qquad (D10)$$

Using this operation many times we can join a finite number of classes (sets). But there is an operation allowing to join a set of sets, i.e. to join a finite or infinite number of sets:

U $$\forall X_1\ M(X_1) \rightarrow \exists X_0\ M(X_0) \wedge \forall X_2\ M(X_2) \rightarrow \qquad (A5)$$
$$(X_2 \in X_0 \rightleftarrows \exists X_3\ M(X_3) \wedge X_2 \in X_3 \wedge X_3 \in X_1)$$

The set X_0, joining set X_1 of sets X_3, is unique (for given X_1). It allows to introduce the next definition:

$$X_2 = \cup X_1 \rightleftarrows \forall X_3\ M(X_3) \rightarrow (X_3 \in X_2 \rightleftarrows \exists X_4\ M(X_4) \wedge X_3 \in X_4 \wedge X_4 \in X_1) \qquad (D11)$$

There exists also a set X_0 of all subsets of X_1:

W $$\forall X_1\ M(X_1) \rightarrow \exists X_0\ M(X_0) \wedge \forall X_2\ M(X_2) \rightarrow (X_2 \in X_0 \rightleftarrows X_2 \subseteq X_1) \qquad (A6)$$

For given X_1 the set X_0 is unique. This allows to introduce the next denotation:

$$X_2 = \mathcal{P}(X_1) \rightleftarrows \forall X_3\ M(X_3) \rightarrow (X_3 \in X_2 \rightleftarrows X_3 \subseteq X_1) \qquad (D12)$$

The next axiom is an existence of a set, being the intersection of any class and any set:

S $$\forall X_1\ M(X_1) \rightarrow \forall X_2 \exists X_0\ M(X_0) \wedge \forall X_3\ M(X_3) \rightarrow (\ X_3 \in X_0 \rightleftarrows X_3 \in X_1 \wedge X_3 \in X_2\ ) \qquad (A7)$$

So the intersection of any set and any class is a set.

The last axiom is existence of transfinite sets constructed from the empty set:

I $$\exists X_0\ M(X_0) \wedge 0 \in X_0 \wedge \forall X_1\ M(X_1) \rightarrow (\ X_1 \in X_0 \rightarrow (X_1 \cup \{X_1\}) \in X_0\ ) \qquad (A8)$$

It follows from this axiom that $0 \in X_0$, $\{0\} \in X_0$, $\{0, \{0\}\} \in X_0$, $\{0, \{0\}, \{0, \{0\}\}\} \in X_0$ etc. This means the infinite set ω of all such members exists and $\omega \cup \{\omega\}$ exists too, etc.

We exclude the axiom of extensionality, since it is a valid formula in logic with equality.

2. Multi-Place Classes.

The multi-place classes contain construction $< X_1, X_2 ..., X_n >$, called n-tuples ($n > 1$), X_1, X_2,..., X_n are called elements of the n-tuple. At $n = 2$ these constructions are called (ordered) pairs, at n=3 - triples.

All n-tuples at $n > 2$ have a recursive definition:

$$< X_1, X_2, ..., X_{n+1} > = << X_1, X_2, ..., X_n >, X_{n+1} > \qquad (D13)$$

Every n-tuple is a set if every pair is a set.

Intuitively, the pairs are not sets. So the definition of the pair is artificial in NBG and has no intuitive meaning:

$$< X_1, X_2 > = \{\{X_1\}, \{X_1, X_2\}\} \qquad (D14)$$

But this definition ensures the basic property of all pairs:

$$< X_1, X_2 > = < X_3, X_4 > \rightarrow X_1 = X_3 \wedge X_2 = X_4$$

The additional properties of pairs are put by the next axioms:

B1 $$\exists X_0 \forall X_1 \forall X_2\ M(X_1) \wedge M(X_2) \rightarrow (< X_1, X_2 >\in X_0 \rightleftarrows X_1 \in X_2) \qquad (A9)$$

B4 $$\forall X_1 \exists X_0 \forall X_2\ M(X_2) \rightarrow (X_2 \in X_0 \rightleftarrows \exists X_3\ M(X_3) \wedge < X_2, X_3 >\in X_1) \qquad (A10)$$

B5 $$\forall X_1 \exists X_0 \forall X_2 \forall X_3\ M(X_2) \wedge M(X_3) \rightarrow (< X_2, X_3 >\in X_0 \rightleftarrows X_2 \in X_1) \qquad (A11)$$

It follows from axiom (A9) and (A10), there are classes of pairs corresponding to relations " $\in$ " and "domain". The axiom (A11) constructs pairs with the second fictive elements.

According to axiom (A10) there exist a class X_0 that consist of the first elements of pairs if these pairs belong to class X_1. So we can define a class $\mathcal{D}(X_1)$, that is domain $\mathcal{D}$ of X_1, if X_1 contains pairs:

$$\mathcal{D}(X_1) = X_2 \rightleftarrows \forall X_3\ M(X_3) \rightarrow (\ X_3 \in X_2 \rightleftarrows \exists X_4\ M(X_4) \wedge < X_3, X_4 >\in X_1\ ) \qquad (D15)$$

We can also define a class, which is a range $\mathcal{R}$ of X_1, if X_1 contains pairs:

$$\mathcal{R}(X_1) = X_2 \rightleftarrows \forall X_3\ M(X_3) \rightarrow (\ X_3 \in X_2 \rightleftarrows \exists X_4\ M(X_4) \wedge < X_4, X_3 >\in X_1) \qquad (D16)$$

Range $\mathcal{R}$ consists of the second elements of the pairs of X_1.

The next two axioms put permutable properties of triples.

B6 $$\forall X_1 \exists X_0 \forall X_2 \forall X_3 \forall X_4\ M(X_2) \wedge M(X_3) \wedge M(X_4) \rightarrow (< X_2, X_3, X_4 >\in X_1 \rightleftarrows < X_3, X_4, X_2 >\in X_0) \qquad (A12)$$

B7 $$\forall X_1 \exists X_0 \forall X_2 \forall X_3 \forall X_4\ M(X_2) \wedge M(X_3) \wedge M(X_4) \rightarrow (< X_2, X_3, X_4 >\in X_1 \rightleftarrows < X_2, X_4, X_3 >\in X_0) \qquad (A13)$$

It is possible also to show, there exist classes X_1, that consist of pairs with a functional dependence of the second element from the first one. So we can define a relation Un, which selects such classes:

$$Un(X_1) \rightleftarrows \forall X_2 \forall X_3 \forall X_4\ M(X_2) \wedge M(X_3) \wedge M(X_4) \wedge < X_2, X_3 >\in X_1 \wedge < X_2, X_4 >\in X_1 \rightarrow X_3 = X_4 \qquad (D17)$$

The class X_1, selected by relation Un, has the important property - if some subclass of domain of X_1 is a set, then corresponding subclass of range of X_1 is a set too:

R $$\forall X_1 \forall X_2\ M(X_2) \wedge Un(X_1) \rightarrow \exists X_0\ M(X_0) \wedge \forall X_3\ M(X_3) \rightarrow (\ X_3 \in X_0 \rightleftarrows \exists X_4\ M(X_4) \wedge < X_4, X_3 >\in X_1 \wedge X_4 \in X_2\ ) \qquad (A14)$$

This axiom is the last in NBG-theory.

3. First Normal Form for One-Place Classes.

To reduce to the first normal form, sentences must be in the $\forall\exists$-prenex form. Variables bound by universal quantifiers are denoted by x_1, x_2,... . Variables bound by existential quantifiers are denoted by a_1, a_2,... .

As an exception, we can use sentences in $\exists\forall$-prenex form. In this case variables bound by existential quantifiers are denoted by c_1, c_2,... .

In the other case we should enter the new relations which definition absorbs superfluous quantifiers. Below we denote these relations by W1, W2,... .

So reducing to the first normal form we have next clauses:

1. $x_1 = x_2 \ \vee\ a_1 \in x_1 \wedge a_1 \notin x_2 \ \vee\ a_1 \notin x_1 \wedge a_1 \in x_2$ (definition of "=").
2. $x_1 \subseteq x_2 \ \vee\ a_1 \in x_1 \wedge a_1 \notin x_2$ (definition of subset).
3. $x_1 \nsubseteq x_2 \ \vee\ x_3 \notin x_1 \ \vee\ x_3 \in x_2$.
4. $M(x_1) \ \vee\ x_1 \notin x_2$ (definition of set).
5. $\neg M(x_1) \ \vee\ x_1 \in a_1$.
6. $M(0)$ (axiom of 0).
7. $0 = x_1 \ \vee\ a_1 \in x_1$.
8. $x_1 \notin 0$.
9. $M\{x_1, x_2\} \ \vee\ \neg M(x_1) \ \vee\ \neg M(x_2)$ (axiom of unordered pair).
10. $\{x_1, x_2\} = x_3 \ \vee\ M(x_1) \ \vee\ M(x_2) \ \vee\ a_1 \in x_3 \wedge a_1 \neq x_1 \wedge a_1 \neq x_2$.
11. $\{x_1, x_2\} = x_3 \ \vee\ M(x_1) \ \vee\ x_2 \notin x_3 \ \vee\ a_1 \in x_3 \wedge a_1 \neq x_1 \wedge a_1 \neq x_2$.
12. $\{x_1, x_2\} = x_3 \ \vee\ x_1 \notin x_3 \ \vee\ M(x_2) \ \vee\ a_1 \in x_3 \wedge a_1 \neq x_1 \wedge a_1 \neq x_2$.
13. $\{x_1, x_2\} = x_3 \ \vee\ x_1 \notin x_3 \ \vee\ x_2 \notin x_3 \ \vee\ a_1 \in x_3 \wedge a_1 \neq x_1 \wedge a_1 \neq x_2$.
14. $x_1 \in \{x_1, x_2\} \ \vee\ \neg M(x_1)$.
15. $x_2 \in \{x_1, x_2\} \ \vee\ \neg M(x_2)$.
16. $x_3 \notin \{x_1, x_2\} \vee x_3 = x_1 \ \vee\ x_3 = x_2$.
17. $\{x_1\} = \{x_1, x_1\}$ (definition of one-member set).
18. $x_1 \cap x_2 = x_3 \ \vee\ a_1 \in x_3 \wedge a_1 \notin x_1 \ \vee\ a_1 \in x_3 \wedge a_1 \notin x_2 \ \vee\ a_1 \notin x_3 \wedge a_1 \in x_1 \wedge a_1 \in x_2$ (axiom of set intersection).

19. $x_3 \in x_1 \cap x_2 \ \vee \ x_3 \notin x_1 \ \vee \ x_3 \notin x_2$.
20. $x_3 \notin x_1 \cap x_2 \ \vee \ x_3 \in x_1$.
21. $x_3 \notin x_1 \cap x_2 \ \vee \ x_3 \in x_2$.
22. $\overline{x}_1 = x_2 \ \vee \ a_1 \in x_1 \wedge a_1 \in x_2 \ \vee \ M(a_1) \wedge a_1 \notin x_1 \wedge a_1 \notin x_2$ (axiom of complement).
23. $x_2 \notin \overline{x}_1 \ \vee \ x_2 \notin x_1$.
24. $\neg M(x_2) \ \vee \ x_2 \in \overline{x}_1 \ \vee \ x_2 \in x_1$.
25. $x_1 \cup x_2 = \overline{\overline{x}_1 \cap \overline{x}_2}$ (definition of set join).
26. $M(\cup x_1) \ \vee \ \neg M(x_1)$ (axiom of join for set of sets).
27. $\cup x_1 = x_2 \ \vee \ a_1 \notin x_2 \wedge \mathrm{W1}(x_1, a_1) \ \vee \ a_1 \in x_2 \wedge \neg \mathrm{W1}(x_1, a_1)$.
28. $\mathrm{W1}(x_1, x_2) \ \vee \ x_2 \notin x_3 \ \vee \ x_3 \notin x_1$.
29. $\neg \mathrm{W1}(x_1, x_2) \ \vee \ x_2 \in a_1 \wedge a_1 \in x_1$.
30. $x_2 \in \cup x_1 \ \vee \ x_2 \notin x_3 \ \vee \ x_3 \notin x_1$.
31. $x_2 \notin \cup x_1 \ \vee \ x_2 \in a_1 \wedge a_1 \in x_1$.
32. $M(\mathcal{P}(x_1)) \ \vee \ \neg M(x_1)$ (axiom of power set).
33. $\mathcal{P}(x_1) = x_2 \ \vee \ a_1 \in x_2 \wedge a_1 \nsubseteq x_1 \ \vee \ M(a_1) \wedge a_1 \notin x_2 \wedge a_1 \subseteq x_1$.
34. $x_2 \notin \mathcal{P}(x_1) \ \vee \ x_2 \subseteq \mathcal{P}(x_1)$.
35. $x_2 \in \mathcal{P}(x_1) \ \vee \ \neg M(x_2) \ \vee \ x_2 \nsubseteq x_1$.
36. $M(x_1 \cup x_2) \ \vee \ \neg M(x_1)$ (axiom of intersection of set and class).
37. $M(c_1) \wedge 0 \in c_1 \wedge x_1 \notin c_1 \ \vee \ M(c_1) \wedge 0 \in c_1 \wedge (x_1 \cup \{x_1\}) \in c_1$ (axiom of infinity).

4. First Normal Form for Multi-Place Classes.

Reducing to the first normal form we have next sentences (see denotation in section 3):

38. $< x_1, x_2 >= \{\{x_1\}, \{x_1, x_2\}\}$ (definition of ordered pairs).
39. $< x_1, x_2, x_3 >=<< x_1, x_2 >, x_3 >$.
40. $< x_1, x_2 >\in c_1 \ \vee \ \neg M(x_2) \ \vee \ x_1 \notin x_2$ ($\in$-axiom).
41. $< x_1, x_2 >\notin c_1 \ \vee \ \neg M(x_1) \ \vee \ \neg M(x_2) \ \vee \ x_1 \in x_2$
42. $\mathcal{D}(x_1) = x_2 \ \vee \ M(a_1) \wedge a_1 \notin x_2 \wedge \mathrm{W2}(x_1, a_1) \ \vee \ a_1 \in x_2 \wedge \neg \mathrm{W2}(x_1, a_1)$ (axiom of domain).
43. $\mathrm{W2}(x_1, x_2) \ \vee \ \neg M(x_3) \ \vee \ < x_2, x_3 >\notin x_1$.
44. $\neg \mathrm{W2}(x_1, x_2) \ \vee \ M(a_1) \wedge < x_2, a_1 >\in x_1$.
45. $x_2 \in \mathcal{D}(x_1) \ \vee \ \neg M(x_2) \ \vee \ \neg M(x_3) \ \vee \ < x_2, x_3 >\notin x_1$.
46. $x_2 \notin \mathcal{D}(x_1) \ \vee \ M(a_1) \wedge < x_2, a_1 >\in x_1$.
47. $fict(x_1) = x_2 \ \vee \ M(a_1) \wedge M(a_2) \wedge < a_1, a_2 >\in x_2 \wedge a_1 \notin x_1 \ \vee \ M(a_2) \wedge < a_1, a_2 >\notin x_2 \wedge a_1 \in x_1$ (axiom of second fictive place).
48. $< x_2, x_3 >\in fict(x_1) \ \vee \ \neg M(x_3) \ \vee \ x_2 \notin x_1$.
49. $< x_2, x_3 >\notin fict(x_1) \ \vee \ \neg M(x_3) \ \vee \ x_2 \in x_1$.
50. $circ(x_1) = x_2 \ \vee \ M(a_1) \wedge M(a_2) \wedge M(a_3) \wedge < a_1, a_2, a_3 >\in x_1 \wedge < a_2, a_3, a_1 >\notin x_2 \ \vee \ M(a_1) \wedge M(a_2) \wedge M(a_3) \wedge < a_1, a_2, a_3 >\notin x_1 \wedge < a_2, a_3, a_1 >\in x_2$ (axiom of circular permutation).
51. $\neg M(x_2) \ \vee \ \neg M(x_3) \ \vee \ \neg M(x_4) \ \vee \ < x_2, x_3, x_4 >\notin x_1 \ \vee \ < x_3, x_4, x_2 >\in circ(x_1)$.
52. $\neg M(x_2) \ \vee \ \neg M(x_3) \ \vee \ \neg M(x_4) \ \vee \ < x_2, x_3, x_4 >\in x_1 \ \vee \ < x_3, x_4, x_2 >\notin circ(x_1)$.
53. $last(x_1) = x_2 \ \vee \ M(a_1) \wedge M(a_2) \wedge M(a_3) \wedge < a_1, a_2, a_3 >\in x_1 \wedge < a_1, a_3, a_2 >\notin x_2 \ \vee \ M(a_1) \wedge M(a_2) \wedge M(a_3) \wedge < a_1, a_2, a_3 >\notin x_1 \wedge < a_1, a_3, a_2 >\in x_2$ (axiom of permutation of last two places).
54. $\neg M(x_2) \ \vee \ \neg M(x_3) \ \vee \ \neg M(x_4) \ \vee \ < x_2, x_3, x_4 >\notin x_1 \ \vee \ < x_2, x_4, x_3 >\in last(x_1)$.
55. $\neg M(x_2) \ \vee \ \neg M(x_3) \ \vee \ \neg M(x_4) \ \vee \ < x_2, x_3, x_4 >\in x_1 \ \vee \ < x_2, x_4, x_3 >\notin last(x_1)$.
56. $Un(x_1) \ \vee \ M(a_1) \wedge M(a_2) \wedge M(a_3) \wedge < a_1, a_2 >\in x_1 \wedge < a_1, a_3 >\in x_1 \wedge a_2 \neq a_3$ (definition of functional dependence).
57. $\neg Un(x_1) \ \vee \ \neg M(x_2) \ \vee \ \neg M(x_3) \ \vee \ \neg M(x_4) \ \vee \ < x_2, x_3 >\notin x_1 \ \vee \ < x_2, x_4 >\notin x_1 \ \vee \ x_3 = x_4$.
58. $M(repl(x_1, x_2)) \vee \neg Un(x_1) \vee \neg M(x_2)$ (axiom of replacement).
59. $repl(x_1, x_2) = x_3 \ \vee \ M(a_1) \wedge a_1 \notin x_3 \wedge \mathrm{W3}(x_1, x_2, a_1) \ \vee \ a_1 \in x_3 \wedge \neg \mathrm{W3}(x_1, x_2, a_1)$.
60. $\mathrm{W3}(x_1, x_2, x_3) \ \vee \ \neg < x_4, x_3 >\notin x_1 \ \vee \ x_4 \notin x_2$.
61. $\neg \mathrm{W3}(x_1, x_2, x_3) \ \vee \ < a_1, x_3 >\in x_1 \wedge a_1 \in x_2$.
62. $x_3 \in repl(x_1, x_2) \ \vee \ \neg M(x_3) \ \vee \ < x_4, x_3 >\notin x_1 \ \vee \ x_4 \notin x_2$.
63. $x_3 \notin repl(x_1, x_2) \ \vee \ < a_1, x_3 >\in x_1 \wedge a_1 \in x_2$.

That is all. We have many axioms but without quantifiers. This set of axioms is a program ready for the execution.

References

[1]. E. Mendelson. *Introduction to mathematical logic.* Nostrand company (1964).

[2]. M.A. Malkov. Introduction to relational logic. *Relational logic*, 2001, 1.